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Optimal robust allocation of private goods*

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Abstract We characterize the optimal robust mechanisms for the allocation of private objects, where robust mechanisms are those mechanisms that satisfy dominant strategy incentive compatibility, ex-post individual rationality, and ex-post no budget deficit, and optimal robust mechanisms are the ones that maximize the expected sum of players' payoffs among all robust mechanisms. With a certain assumption on the payoff of the lowest possible type, we provide a complete description of optimal robust mechanisms with any number of players and objects.

Keywords robust mechanism design, dominant strategy, budget balance, ex-post individual rationality

JEL Classification C72, D47, D82

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1. INTRODUCTION

We study the optimal allocation of private objects among many players. In particular, we are interested in the mechanisms that satisfy dominant strategy incentive compatibility, ex-post individual rationality, and ex-post no budget deficit. We call this family of mechanisms as robust mechanisms since they do not depend on the fine details of players' beliefs.[1] We characterize the optimal robust mechanisms that maximize the expected sum of players' payoffs among all robust mechanisms. With a certain assumption on the payoff of the lowest possible type, we provide a complete description of optimal mechanisms with any number of players and objects.

Drexl and Kleiner (2015) and Shao and Zhou (2016) have analyzed similar problems, with the additional restrictions that (i) there are only two players and one object, (ii) the mechanisms are deterministic, and (iii) the object should always be assigned to one of the players. Their main finding is, roughly speaking, that the optimal mechanism is either a posted price mechanism or an option mechanism when the hazard rate of players' type distributions that the mechanism designer perceives is weakly increasing.[2] In particular, ex-post budget balance is satisfied in the optimal mechanism.[3]

We obtain a similar result as a corollary of our main characterization, without the aforementioned restrictions. The price to be paid for this generalization is an assumption that essentially sets the payoff of the lowest possible type to zero. We note that this assumption is also employed by Kuzmics and Steg (2017) when characterizing deterministic robust mechanisms for public good provision among many players. Hence, it appears to be a crucial condition when dealing with more than two players.

Other related literature includes Hagerty and Rogerson (1987) and Čopič and Ponsatí (2016), who characterize robust mechanisms for the bilateral trading setting in which two players, a seller and a buyer, bargain over one private object: They show that optimal robust mechanisms are posted price mechanisms, with the ex-post budget balance condition imposed a priori instead of the weaker ex-

[1] There is a voluminous literature on the informational robustness of mechanisms to the common knowledge assumption concerning players' beliefs about each other, after the pioneering work of Bergemann and Morris (2005).

[2] Please refer to the next section for details. There are small but significant differences between these papers.

[3] Ex-post no budget deficit is satisfied if the mechanism does not run a budget deficit in any realization, whereas ex-post budget balance is satisfied if it runs neither a budget deficit nor a budget surplus in any realization.

post no budget deficit condition. We emphasize that, compared to the papers cited above, only the present paper examines robust mechanisms for the allocation of private objects with any number of players and objects.

In the next section, we first consider the one object case and characterize optimal robust mechanisms (Proposition 1). One of the implications is that it is optimal to assign the object with probability one: Note that this property is endogenously derived in the present paper, not exogenously assumed as in Drexl and Kleiner (2015) and Shao and Zhou (2016). This is in nice contrast with Guo and Conitzer (2014) and de Clippel *et al.* (2014) who show that it is optimal to sometimes destroy the object under the worst-case optimality criterion. Corollaries 1, 2 and the subsequent examples further characterize optimal robust mechanisms. Careful comparison with the existing literature is given. We then extend the analysis to the case of multiple, possibly heterogeneous, objects when each player demands at most one object (Proposition 2).

The analysis in this paper is rather standard. With an appropriate framework, we not only derive the main results straightforwardly but also provide a useful perspective on the existing results, in particular, clarify the assumptions that are needed for the results. Similar techniques are used in Hartline and Roughgarden (2008), Yoon (2011), Condorelli (2012), and Chakravarty and Kaplan (2013) for the Bayesian analysis of the allocation problem with socially costly expenditures. In comparison, we study dominant strategy incentive compatibility and the expenditures may not be socially costly.

2. OPTIMAL ROBUST MECHANISMS

2.1. MAIN RESULTS

We consider a situation where one object is to be assigned to one of the players in $N = \{1,\ldots,n\}$. Each player $i \in I$ has a valuation v_i for the object. We assume that this is private information. The set of player i's possible valuations is given as $V_i = [\underline{v}_i, \overline{v}_i]$, with $0 = \underline{v}_i \leq \overline{v}_i \leq \infty$. We set $\underline{v}_i = 0$ to align with most papers in this literature: We note that it does not cause any challenge to the analysis or meaningful change to the results to set $\underline{v}_i > 0$ in this paper. We use usual notation such as $v = (v_1,\ldots,v_n)$, $V = V_1 \times \cdots \times V_n$, $v_{-i} = (v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_n)$ and $V_{-i} = \times_{j \neq i} V_j$. We also use $v = (v_i, v_{-i})$ when we focus on i's perspective.

A direct mechanism is a pair (p,x), with $p : V \to [0,1]^n$ and $x : V \to \Re^n$. Hence, if the reported vector of valuations is v', then $p_i(v')$ is the assignment probability that i gets the object and $x_i(v')$ is the expenditure that i exerts. The expenditure is usually a monetary transfer made by i, but may also be a physical expenditure such as effort. Player i's payoff when his true valuation is v_i and the reported vector of valuations is v' is given as $p_i(v')v_i - x_i(v')$. Define

$$u_i(v) = p_i(v)v_i - x_i(v).$$

By the revelation principle, it is with no loss of generality to restrict our attention to direct mechanisms.

The mechanism (p,x) is said to satisfy *dominant strategy incentive compatibility*[4] if

$$u_i(v_i,v_{-i}) \geq p_i(v_i',v_{-i})v_i - x_i(v_i',v_{-i}),\ \ \forall i\in N, v_i, v_i' \in V_i, v_{-i}\in V_{-i}, \qquad (IC)$$

and *ex-post individual rationality* if

$$u_i(v_i,v_{-i}) \geq 0,\ \ \forall i\in N,\ \forall v_i\in V_i, v_{-i}\in V_{-i}. \qquad (IR)$$

The mechanism (p,x) is said to satisfy the *ex-post no deficit condition* if

$$\sum_{i=1}^{n} x_i(v) \geq 0,\ \ \forall v\in V, \qquad (ND)$$

and the probability condition if, in addition to the obvious requirement of $p_i(v)\geq 0$ for all $i\in N$ and $v\in V$,

$$\sum_{i=1}^{n} p_i(v) \leq 1,\ \ \forall v\in V. \qquad (P)$$

We call the mechanism (p,x) to be *robust* if it satisfies (IC), (IR) and (ND) in addition to (P).

We have the following lemma. We omit the proof since it is standard.

Lemma 1. *The mechanism (p,x) satisfies (IC) and (IR) if and only if*

$$v_i \leq v_i' \Rightarrow p_i(v_i,v_{-i}) \leq p_i(v_i',v_{-i})\ \ \forall i\in N, v_i, v_i'\in V_i, v_{-i}\in V_{-i}, \qquad (M)$$

$$u_i(v) = \int_0^{v_i} p_i(w_i,v_{-i})dw_i + u_i(0,v_{-i})\ \ \forall i\in N, v_i\in V_i, v_{-i}\in V_{-i}, \qquad (I)$$

$$u_i(0,v_{-i}) \geq 0\ \forall i\in N, v_{-i}\in V_{-i}. \qquad (L)$$

[4]Dominant strategy incentive compatibility is also termed truthfulness or strategyproofness.

The first property (M) is monotonicity: player i's assignment probability is weakly increasing in v_i. The second property (I) is integrability, and the last property (L) is the ex-post individual rationality for the lowest possible type. Our objective is to characterize the optimal robust mechanisms that maximize the (ex-ante) expected sum of players' payoffs among all robust mechanisms. Let us assume that the mechanism designer has a subjective belief $F(v)$ about players' valuation vectors, and moreover, $F(v) = F_1(v_1) \times \cdots \times F_n(v_n)$ where $F_i(v_i)$ is the distribution of player i's valuation. Hence, we assume independence. Let $f_i(v_i)$ be the corresponding density function and let $f(v) = f_1(v_1) \times \cdots \times f_n(v_n)$. Note well that the players need not have any belief, let alone their beliefs are common. That is, F pertains only to the designer.

It is sometimes desirable that the mechanism satisfies the *ex-post budget balance condition*

$$\sum_{i=1}^{n} x_i(v) = 0, \;\; \forall v \in V, \tag{BB}$$

and/or the *no waste condition*

$$\sum_{i=1}^{n} p_i(v) = 1, \;\; \forall v \in V. \tag{NW}$$

The ex-post budget balance condition is desirable when the designer does not value the expenditure, and the no waste condition is desirable when the designer does not value the object. Note that (BB) is stronger than (ND) and (NW) is stronger than (P).

Define

$$q_i(v_i) = \int_{V_{-i}} p_i(v) f_{-i}(v_{-i}) dv_{-i} \text{ and}$$

$$y_i(v_i) = \int_{V_{-i}} x_i(v) f_{-i}(v_{-i}) dv_{-i},$$

where $f_{-i}(v_{-i})$ is the joint density of v_{-i}. Thus, $q_i(v_i)$ is player i's conditional expected assignment probability and $y_i(v_i)$ is player i's conditional expected expenditure. Since

$$x_i(v) = p_i(v)v_i - \int_0^{v_i} p_i(w_i, v_{-i}) dw_i + x_i(0, v_{-i})$$

by definition of $u_i(v)$ and property (I) of Lemma 1, we have

$$y_i(v_i) = q_i(v_i)v_i - \int_0^{v_i} q_i(w_i) dw_i + y_i(0).$$

Hence, the designer's objective is to maximize

$$\int_V \Big(\sum_{i=1}^n [p_i(v)v_i - x_i(v)]\Big) f(v)dv = \sum_{i=1}^n \int_0^{\bar{v}_i} [q_i(v_i)v_i - y_i(v_i)] f_i(v_i)dv_i$$

$$= \sum_{i=1}^n \int_0^{\bar{v}_i} \Big[\int_0^{v_i} q_i(w_i)dw_i - y_i(0)\Big] f_i(v_i)dv_i$$

$$= \int_V \Big(\sum_{i=1}^n \frac{1-F_i(v_i)}{f_i(v_i)} p_i(v)\Big) f(v)dv - \sum_{i=1}^n y_i(0)$$

subject to (M), (L), (ND) and (P).

Let us assume that $\sum_{i=1}^n y_i(0) \geq 0$, that is, the sum of the conditional expected expenditure of the lowest possible type is not less than zero. This assumption holds, for instance, when x_i is physical expenditure and so negative values are impossible a priori.[5] It may also hold even when $x_i(v) < 0$ for some $i \in N$ and $v \in V$. We may alternatively impose that $x_i(0, v_{-i}) \geq 0$ for all i and $v_{-i} \in V_{-i}$, i.e., any player's expenditure is always greater than or equal to zero when his type is the lowest possible type, as a requirement of the mechanism: The famous VCG (Vickrey-Clarke-Groves) mechanism and the random allocation mechanism satisfy this requirement, for instance. As a matter of fact, both the VCG mechanism and the random allocation mechanism satisfy this requirement with equality. Note that this is weaker than the requirement of $x_i(0, v_{-i}) = 0$ for all i and $v_{-i} \in V_{-i}$ imposed by Guo and Conitzer (2014) for their 'linear' allocation mechanisms. See also Morimoto and Serizawa (2015) for related conditions of no subsidy and no subsidy for losers: No subsidy requires $x_i(v_i, v_{-i}) \geq 0$ for all i, v_i, and v_{-i} whereas no subsidy for losers requires $x_i(v_i, v_{-i}) \geq 0$ when $p_i(v_i, v_{-i}) = 0$. However, the assumption we impose is certainly restrictive and we will discuss it more carefully after Corollary 2 below. Anyway, with this assumption together with the property (L) in Lemma 1, it is optimal to set $x_i(0, v_{-i}) = 0$ and so $y_i(0) = 0$ for the maximization. Hence, the designer's problem is to choose the assignment probabilities p to maximize

$$\int_V \Big(\sum_{i=1}^n \frac{1-F_i(v_i)}{f_i(v_i)} p_i(v)\Big) f(v)dv$$

[5] This is the case in the works of Hartline and Roughgarden (2008), Yoon (2011), Condorelli (2012), and Chakravarty and Kaplan (2013).

subject to (M), (ND) and (P).

Before presenting the main results, we remind the reader of a well-known fact. A random variable X or its distribution F is said to be IFR if the hazard rate (i.e., the failure rate) $f(x)/(1-F(x))$ is weakly increasing, and DFR if the hazard rate is weakly decreasing. Equivalently, it is IFR (DFR) if the survival rate $1-F(x)$ is log-concave (log-convex). Examples of IFR distributions are exponential, uniform, normal, logistic, power (for $b \geq 1$), Weibull (for $b \geq 1$), and gamma (for $b \geq 1$), while those of DFR distributions are exponential, Weibull (for $0 < b \leq 1$), gamma (for $0 < b \leq 1$), and Pareto.[6]

To characterize optimal robust mechanisms, define the functions from $[0,1]$ to $\Re$ as follows:

$$h_i(q) = \tfrac{1-q}{f_i(F_i^{-1}(q))},\ H_i(q) = \textstyle\int_0^q h_i(r)dr,$$

$$G_i(q) = \text{conv}H_i(q),\ g_i(q) = G_i'(q).$$

Note that G_i is the convex hull of the function H_i, that is, the highest convex function with $G_i(q) \leq H_i(q)$ for all $q \in [0,1]$. As a convex function, G_i is continuously differentiable except at countably many points, and its derivative is monotone increasing. Define g_i as extending by right-continuity when G_i is not differentiable. Finally, let

$$c_i(v_i) = g_i(F_i(v_i)) \ \text{ and } \ M(v) = \{i \in N | c_i(v_i) = \max_{j \in N} c_j(v_j)\}.$$

We have:

Proposition 1. *Assume that $\sum_{i=1}^n y_i(0) \geq 0$. Let $p : V \to [0,1]^n$ and $x : V \to \Re^n$ satisfy*

$$p_i(v) = \begin{cases} 1/|M(v)| & \text{if } i \in M(v) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x_i(v) = p_i(v)v_i - \int_0^{v_i} p_i(w_i, v_{-i})dw_i.$$

Then, (p,x) represents an optimal robust mechanism.

Note that we use $|S|$ to denote the cardinality of an arbitrary set S. We omit the proof since it is a simple adaptation of that in Myerson (1981). Kuzmics and Steg (2017) study a public good provision problem and characterize the

[6] Here, b is the shape parameter.

deterministic direct mechanisms which satisfy (IC), (IR) and (BB).[7] One of their main results is that (i) A deterministic direct mechanism satisfies (IC), (IR) and (BB) together with either $u_i(\underline{v}_i, v_{-i}) = 0$ for all $i \in N$ and $v_{-i} \in V_{-i}$ or cost sharing if and only if it is a threshold mechanism. They also show that (IC), (BB) and the condition $u_i(\underline{v}_i, v_{-i}) = 0$ for $i \in N$ and $v_{-i} \in V_{-i}$ imply cost sharing. Our result is a similar characterization for the private good allocation problem. Note, however, that Kuzmics and Steg (2017) characterize all robust mechanisms whereas we characterize optimal robust mechanisms. One of the reasons for this difference is that, besides the fact that they restrict attention only to deterministic mechanisms, the public good problem is much simpler in the sense that the allocation rule in the public good problem is a single decision variable, say $r : V \to \{0,1\}$, while that in the private good problem is $p_i : V \to [0,1]^n$ for all $i \in N$. Note in particular that $r(v)$ is weakly increasing in v_j for every $j \in N$ in the public good problem since every player faces the same decision. In contrast, we can only guarantee that $p_i(v)$ for each $i \in N$ is weakly increasing in v_i in the private good allocation problem.

Another observation worth mentioning is that the optimal assignment rule satisfies the no waste condition (NW). Guo and Conitzer (2014) and de Clippel *et al.* (2014) show that it may be better that the object is not always assigned under some circumstances: They employ the worst-case optimality criterion and show that it is optimal to sometimes destroy the object. In contrast, we show that it is optimal to assign the object with probability 1, i.e., (NW) is satisfied, when we maximize the (ex-ante) expected sum of players' payoffs.

We consider some special cases. First, assume that all F_i's for $i \in N$ are DFR. Then, $(1 - F_i(v_i))/f_i(v_i)$ in the maximization problem is weakly increasing, and we have $G_i = H_i$, $g_i = h_i$, and $c_i(v_i) = (1 - F_i(v_i))/f_i(v_i)$. Hence, the designer's objective is maximized by assigning the object to any one of the players with the highest $(1 - F_i(v_i))/f_i(v_i)$. Summarizing,

Corollary 1. *Assume that $\sum_{i=1}^n y_i(0) \geq 0$. Assume also that all F_i's for $i \in N$ are DFR. An optimal robust mechanism assigns the object with probability 1 to the player(s) with the highest $(1 - F_i(v_i))/f_i(v_i)$. When players are symmetric so that $F_1 = F_2 = \cdots = F_n$, an optimal robust mechanism assigns the object with probability 1 to the player(s) with the highest v_i.*

[7] A deterministic direct mechanism has the property that the assignment rule takes only the value zero or one. We note that Kuzmics and Steg (2017) call the ex-post no deficit condition (ND) as ex-post budget balance and the ex-post budget balance condition (BB) as exact ex-post budget balance.

Next, assume that all F_i's for $i \in N$ are IFR. Then, $h_i(q)$ is weakly decreasing in q and so $H_i(q)$ is a concave increasing function, with $H_i(0) = 0$. Hence, $G_i(q) = H_i(1)q$, a straight line connecting $(0,0)$ and $(1,H(1))$, and we have $g_i(q) = H_i(1)$ and $c_i(v_i) = H_i(1)$. Observe that $H_i(1) = E[v_i]$ since

$$H_i(1) = \int_0^1 \frac{1-r}{f_i(F_i^{-1}(r))} dr = \int_0^{\bar{v}_i} \frac{1-F_i(v_i)}{f_i(v_i)} dF_i(v_i) = E[v_i],$$

where the last equality follows from integration by parts. Hence, the designer's objective is maximized by assigning the object to any one of the players with the highest $E[v_i]$, irrespective of the v_i actually realized. Observe also that we have $x_i(v) = 0$ for all $i \in N$, and hence (BB) is satisfied, when the distribution is IFR. Summarizing,

Corollary 2. *Assume that $\sum_{i=1}^n y_i(0) \geq 0$. Assume also that all F_i's for $i \in N$ are IFR.*

(i) An optimal robust mechanism assigns the object with probability 1 to the player(s) with the highest $E[v_i]$. When players are symmetric so that $F_1 = F_2 = \cdots = F_n$, an optimal robust mechanism assigns the object to each player with equal probability.

(ii) The ex-post budget balance (BB) is satisfied in the optimal robust mechanism.

Drexl and Kleiner (2015) and Shao and Zhou (2016) study the *two-player* optimal private good allocation problem. Drexl and Kleiner (2015) consider deterministic mechanisms that satisfy (IC), (IR), (ND) and (NW). They show that, when the distributions are IFR, the optimal robust mechanism is either a posted price or an option mechanism. In particular, ex-post budget balance (BB) is satisfied in the optimal robust mechanism. Shao and Zhou (2016) obtain a similar result: They consider deterministic mechanisms that satisfy (IC), (ND) and (NW) with the assumption that both V_1 and V_2 are the unit interval $[0,1]$ and show that, when the distributions are both IFR and DRFR,[8] the optimal robust mechanism is either a posted price or an option mechanism. Hence, (BB) is satisfied in the optimal robust mechanism. That is, they drop (IR) but need the DRFR condition in comparison to Drexl and Kleiner (2015).

Corollary 2 obtains a similar result. We do not impose (NW) a priori but derive it endogenously. We also consider general assignment rules, not just deterministic assignment rules. However, we impose the assumption that $\sum_{i=1}^n y_i(0) \geq$

[8]A random variable X or its distribution F is said to be DRFR if the reversed failure rate $f(x)/F(x)$ is weakly decreasing.

0. Note that the two-player case is special since $p_1(v)+p_2(v)=1$ always holds with the no waste condition (NW). That is, it is analytically close to the public good provision problem of determining a single decision variable since we know $p_2(v)$ once we know $p_1(v)$. In contrast, we consider the many-player optimal private good allocation problem, which makes the analysis harder.

The assumption that $\sum_{i=1}^n y_i(0) \geq 0$ is restrictive. To see this, consider a posted price mechanism for two players which by default assigns the object to player 1 and changes the assignment if and only if both players agree to trade at a prespecified price π. Let $0<\pi<\overline{v}_2$. This mechanism satisfies (IC), (IR), (BB) and (NW), but we have $y_1(0)=-\pi(1-F_2(\pi))<0$ and $y_2(0)=0$. Thus, our assumption is violated. Note, however, that this mechanism is asymmetric across players. This implies that a certain symmetry assumption would exclude this mechanism from consideration.

Note that this assumption together with the ex-post individual rationality condition pins down $u_i(0,v_{-i})$ to zero for all $i\in N$ and $v_{-i}\in V_{-i}$. This seems to be the price to be paid when we deal with more than two players since, as discussed above, this assumption is also needed for the main characterization of the public good provision mechanisms in Kuzmics and Steg (2017).

We next show via a series of examples that generally optimal robust mechanisms may take various forms when the distribution is neither IFR nor DFR.

Example 1. There are two players. Let $F_i(v_i)=\sqrt{v_i}/2$ on the support $[\underline{v}_i,\overline{v}_i]=[0,4]$ for $i=1,2$. Note that this distribution is neither IFR nor DFR. Then, $(1-F_i(v_i))/f_i(v_i)=4\sqrt{v_i}-2v_i$, $h_i(q)=8q(1-q)$ and $H_i(q)=4q^2-8q^3/3$ on $[0,1]$. Thus,

$$G_i(q)=\begin{cases} 4q^2-\frac{8}{3}q^3 & \textit{when } 0\leq q\leq \frac{1}{4};\\ -\frac{1}{6}+\frac{3}{2}q & \textit{when } \frac{1}{4}\leq q\leq 1,\end{cases}$$

$$g_i(q)=\begin{cases} \textit{8q(1-q)} & \textit{when } 0\leq q\leq \frac{1}{4};\\ \frac{3}{2} & \textit{when } \frac{1}{4}\leq q\leq 1,\end{cases}$$

$$c_i(v_i)=\begin{cases} 4\sqrt{v_i}-2v_i & \textit{when } 0\leq v_i\leq \frac{1}{4};\\ \frac{3}{2} & \textit{when } \frac{1}{4}\leq v_i\leq 4.\end{cases}$$

When one of the players has a valuation v_i of less than $1/4$, it is optimal to assign the object to the player with the higher v_i. On the other hand, when both players have valuations greater than or equal to $1/4$, it is optimal to assign the object to either player with equal probability.

Example 2. There are three players. Let $F_i(v_i) = \sqrt{v_i}/2$ on the support $[\underline{v}_i, \overline{v}_i] = [0,4]$ for $i = 1,2$ and $F_3(v_3) = v_3/2$ on the support $[0,2]$. Thus, $c_1(v_1)$ and $c_2(v_2)$ are given as in the previous example, and $c_3(v_3) = 1$ for all $v_3 \in [0,2]$. When $\max\{v_1, v_2\} < (3-2\sqrt{2})/2$, it is optimal to assign the object to player 3 regardless of the players' valuations (even when player 3 has the lowest valuation.) On the other hand, when $\max\{v_1, v_2\} > (3-2\sqrt{2})/2$, it is optimal to never assign the object to player 3 (even when player 3 has a higher valuation than players 1 and 2.)

Example 3. There are three players. Let $F_i(v_i) = \sqrt{v_i}/2$ on the support $[\underline{v}_i, \overline{v}_i] = [0,4]$ for $i = 1,2$ and $F_3(v_3) = v_3/3$ on the support $[0,3]$. Thus, $c_1(v_1)$ and $c_2(v_2)$ are given as in the previous examples, and $c_3(v_3) = 3/2$ for all $v_3 \in [0,3]$. When $\max\{v_1, v_2\} < 1/4$, it is optimal to assign the object to player 3. When $\min\{v_1, v_2\} \geq 1/4$, it is optimal to assign the object to players 1, 2 or 3 with equal probability. When $v_1 < 1/4$ and $v_2 \geq 1/4$, it is optimal to assign the object to player 2 or 3 with equal probability. When $v_1 \geq 1/4$ and $v_2 < 1/4$, it is optimal to assign the object to player 1 or 3 with equal probability.

2.2. AN EXTENSION TO MULTIPLE HETEROGENEOUS OBJECTS

It is only a matter of additional notation to extend the analysis to multiple, possibly heterogeneous, objects: All we need to modify is the probability condition of the assignment rule. Let there be m heterogeneous objects to be assigned to the players in N with $m < n$. Assume that each player demands at most one object. Assume also that the objects are commonly valued in that there are $a^1 \geq \cdots \geq a^m > 0$ such that player i's payoff is $\sum_{k=1}^m p_i^k(v')a^k v_i - x_i(v')$ in which $p_i^k(v')$ is the probability that i gets the k-th object and $x_i(v')$ is the expenditure that i makes when the reported vector of valuations is v'. If we define $p_i(v') = \sum_{k=1}^m a^k p_i^k(v')$, then essentially the same analysis applies, with the probability condition (P) adjusted to[9]

$$\sum_{i=1}^n p_i(v) \leq \sum_{k=1}^m a^k, \quad \forall v \in V. \tag{P'}$$

[9]Since each player demands at most one object, we also have $\sum_{k=1}^m p_i^k(v) \leq 1$ for all $i = 1, \ldots, n$ as well as $\sum_{i=1}^n p_i^k(v) \leq 1$ for all $k = 1, \ldots, m$.

Thus, the designer's problem becomes to choose the assignment probabilities p_i^k's to maximize

$$\int_V \Big(\sum_{k=1}^{m} a^k \sum_{i=1}^{n} \frac{1-F_i(v_i)}{f_i(v_i)} p_i^k(v) \Big) f(v) dv$$

subject to (M), (ND) and (P'). It is then optimal to assign the first object to the player with the highest $c_i(v_i)$, the second object to the player with the second highest $c_i(v_i)$ and so on up until all m objects are assigned, with any ties broken evenly. To be precise, define

$$M_1(v) = \{i \in N | c_i(v_i) = \max_{j \in N} c_j(v_j)\}$$

$$M_2(v) = \{i \in N \setminus M_1(v) | c_i(v_i) = \max_{j \in N \setminus M_1(v)} c_j(v_j)\}$$

$$\vdots$$

$$M_l(v) = \{i \in N \setminus (M_1(v) \cup \cdots \cup M_{l-1}(v)) | c_i(v_i) = \max_{j \in N \setminus (M_1(v) \cup \cdots \cup M_{l-1}(v))} c_j(v_j)\}$$

$$\vdots$$

This process ends in a finite step, say step $\bar{l}$, and $\bar{l} \leq n$. We have:

Proposition 2. *Assume that* $\sum_{i=1}^n y_i(0) \geq 0$. *For a player* $i \in M_l(v)$,
(i) If $|M_1(v) \cup \cdots \cup M_{l-1}(v)| \geq m$, *then let* $p_i^k(v) = 0$ *for all* $k = 1, \ldots, m$.
(ii) If $|M_1(v) \cup \cdots \cup M_{l-1}(v)| = k < m$, *then let*

$$p_i^{k+1}(v) = \cdots = p_i^{\min\{k+|M_l(v)|,m\}}(v) = \frac{1}{|M_l(v)|} \text{ and}$$

$$p_i^{k'}(v) = 0 \ \text{ for all } \ k' \in \{1, \ldots, m\} \setminus \{k+1, \ldots, \min\{k+|M_l(v)|, m\}\}.$$

Then, these assignment probabilities $p_i^k(v)$*'s together with*

$$x_i(v) = p_i(v) v_i - \int_0^{v_i} p_i(w_i, v_{-i}) dw_i$$

represent an optimal robust mechanism.

It is obvious that all the subsequent results in the previous subsection carry over without any significant modification.

3. DISCUSSION

We have characterized optimal robust mechanisms for the allocation of private objects among many players, where robust mechanisms are those mechanisms that satisfy dominant strategy incentive compatibility, ex-post individual rationality, and ex-post no budget deficit, and optimal robust mechanisms are the ones that maximize the expected sum of players' payoffs among all robust mechanisms.

The specific form of the optimal robust mechanisms critically depends on how the mechanism designer's subjective belief $F(v) = F_1(v_1) \times \cdots \times F_n(v_n)$ about players' valuation vector behaves. If all F_i's are DFR, an optimal robust mechanism assigns the objects to the player(s) with the highest $(1-F_i(v_i))/f_i(v_i)$'s, and in particular to those with the highest v_i's when players are symmetric. If all F_i's are IFR, an optimal robust mechanism assigns the objects to the player(s) with the highest $E[v_i]$'s, and in particular assigns them randomly when players are symmetric. An optimal robust mechanism may take various forms if F_i's are neither DFR nor IFR. In any case, the optimal robust mechanisms never withhold or destruct the objects. This property is in nice contrast with Guo and Conitzer (2014) and de Clippel *et al.* (2014), as discussed above.

We have shown that budget balance holds in the optimal mechanism when F_i's are IFR. This property is also obtained in the models of Drexl and Kleiner (2015) and Shao and Zhou (2016) who impose the additional restrictions that (i) there are only two players and one object, (ii) the mechanisms are deterministic, and (iii) the object should always be assigned to one of the players.[10] The forms of the optimal robust mechanisms differ, however: It is either a posted price or an option mechanism in their models whereas it is a mechanism that assigns the objects according to $E[v_i]$ in this paper.

Compared to Drexl and Kleiner (2015) and Shao and Zhou (2016), the current paper examines robust mechanisms for the allocation of private goods when there are more than two players as well as when there is more than one object. This was made possible due to our assumption that $\sum_{i=1}^{n} y_i(0) \geq 0$, that is, the sum of the conditional expected expenditure of the lowest possible type is not less than zero. This assumption is reasonable for some situations but certainly restrictive. It remains as a future research agenda whether we can dispense with this assumption.

[10]Note that we derive the last property endogenously whereas they impose it exogenously.

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