

**Optimal quality scores in sponsored search auctions:
Full extraction of advertisers' surplus***

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Abstract

This paper shows that the quality scores in sponsored search auctions can be optimally chosen to extract all the advertisers' surplus. The reason for the full extraction result is that the quality scores may effectively set all the bidders' valuations equal to the highest valuation, which induces intense bidding competition.

Keywords: Online advertising; Sponsored search; Quality score; Full extraction

JEL classifications: D44; M37

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1 Introduction

The sponsored search auction is an innovative trading institution in online advertising. Whenever an Internet user types in a particular search keyword, a new auction is triggered for advertising slots that will display sponsored search results or ads alongside with the organic search results.¹ Advertisers pay to the advertising intermediary only when their ads are clicked. This trading institution has evolved over time since its inception in 1997.² In the current auction format, several advertising slots or positions are simultaneously auctioned off using a payment scheme apparently similar to the second price auction. In particular, each advertiser or bidder pays not his/her own bid but the minimum price that would retain the current position. Edelman *et al.* (2007) call the auction rule in practice as the generalized second-price (GSP) auction.

One of the salient features of this auction is the use of quality scores, which influence the advertisers' positions as well as the minimum bid requirements. As for the positions, the advertisers are not ordered by their bid amounts but by the adjusted bids multiplied by the quality scores. Google initially used the click-through rate to determine the quality score. It later switched to a less transparent system that incorporates such factors as the relevance of the keywords to its ad group, the landing page quality, the advertisers' historical performance, and other relevant factors.³ Yahoo! initially used only the bids to determine the order, but began to use a ranking system similar to Google's in 2007.

The quality score is designed to ensure that the most relevant ads are shown on the

¹ The auction for keywords has started in search advertising, and expanded to contextual advertising on content pages. Though this paper concentrates on the sponsored search auctions, the analysis applies to the wider auctions for keywords.

² For the early history of sponsored search advertising, see Battelle (2005) as well as the papers cited below.

³ The exact formula is not released publicly.

advertising slots. This will generate as many actual clicks as possible, and may help the advertisers and the Internet users as well as Google and Yahoo! The quality score can, in fact, achieve more: This paper shows that, by optimally choosing the quality scores, it is possible to extract all the advertisers' surplus. The reason for the full extraction result is that the quality scores may effectively make advertisers' valuations equal to the highest valuation, thus inducing fierce bidding competition.

The sponsored search auction has recently attracted much academic attention, especially in computer science. The basic properties of sponsored search auctions have been investigated in early papers including Aggarwal *et al.* (2006), Edelman *et al.* (2007), and Varian (2007). The actual practice and evolution of search advertising is nicely presented in Edelman *et al.* (2007), Evans (2008, 2009) and Liu *et al.* (2009). Other notable papers in economics include Athey and Ellison (2007), Börgers *et al.* (2007), and Milgrom (2009).

In the next section, we first describe the auction rule used in the search advertising markets with an emphasis to the quality score. We then list some properties that closely resemble those in Varian (2007). The main result shows that full extraction of advertisers' surplus is always possible when the number of advertisers exceeds the number of advertising slots. Section 3 contains discussion.

2 Main Results

2.1. Preliminaries

Consider a search advertising market organized by a search site (such as Google), a portal (such as Yahoo!), or any entity that acts as an advertising intermediary. We will henceforth call this entity as the auctioneer since auctions have been used in practice. There are K positions (advertising slots) and I bidders (advertisers) with $K \leq I$. Let c^k be the (expected) number of views that an advertisement in position $k = 1, \dots, K$ effectively receives. Without loss of generality, order the positions so that $c^1 > \dots > c^K > 0$. We may also set $c^{K+1} = \dots = c^I = 0$ for analytic convenience. Each bidder is characterized by

two parameters. For $i = 1, \dots, I$, let r_i be the positive click-through rate (CTR for short) or the rate of clicks when viewed, and let v_i be the positive valuation per click. The CTR is related to the relevance of the advertisement with respect to the particular keyword, while the valuation is related to the final payoff resulting from the clicks. Hence, if bidder i is assigned to position k , the (expected) number of clicks is $c^k r_i$ and the total payoff that bidder i obtains is $c^k r_i v_i$. Note that (i) each position's number of views is independent of the bidder, and (ii) each bidder's valuation is independent of the position.⁴

We describe the auction rule used in search advertising markets, which is termed as the generalized second-price auction by Edelman *et al.* (2007), with an emphasis to the quality score. Bidders submit non-negative bids b_i 's per click for their interested keywords. Bid b_i is multiplied by the quality score $q_i > 0$, and these adjusted bids are arranged in a decreasing order.⁵ For this, let $\pi : I \rightarrow I$ be the permutation of bidders according to the order of adjusted bids $q_i b_i$ so that $\pi(k)$ is the bidder with the k -th highest adjusted bid. Then, we have $q_{\pi(1)} b_{\pi(1)} \geq \dots \geq q_{\pi(I)} b_{\pi(I)}$. Bidder $\pi(k)$ for $k = 1, \dots, K$ is assigned to position k , and pays $q_{\pi(k+1)} b_{\pi(k+1)} / q_{\pi(k)}$ per click. That is, a bidder with the highest adjusted bid is assigned to the highest position (position 1), a bidder with the second highest adjusted bid is assigned to the second highest position (position 2) and so on, and then each bidder is charged the minimum price to retain the current position. Note that $b_{\pi(K+1)}$ is well-defined when $K < I$. Otherwise, i.e., when $K = I$, we can let $b_{\pi(K+1)} = 0$.⁶ A bidder in position k has a net payoff or surplus of $c^k r_{\pi(k)} (v_{\pi(k)} - q_{\pi(k+1)} b_{\pi(k+1)} / q_{\pi(k)})$ for $k = 1, \dots, K$. A bidder without a position does not pay and has a surplus of zero.

When $q_i = 1$ for all i , the bidders are arranged according to their submitted bids

⁴ In a more general setting, the total payoff of bidder i who is assigned to position k may be defined as $c_i^k v_i^k$. The value v_i^k may decrease in k if the conversion rate (that is, the rate of transactions/actions when clicked) is declining. The case when $c_i^k = c^k r_i$, as we specify here, is known as the separable or multiplicative form.

⁵ Ties can be broken in any pre-specified way.

⁶ Alternatively, we can set $b_{I+1} = 0$ by convention. Since we have set $c^{K+1} = \dots = c^I = 0$, either convention will do for the following analysis.

and the winning bidders pay the next bid, i.e., bidder $\pi(k)$ pays $b_{\pi(k+1)}$ per click for $k = 1, \dots, K$. This corresponds to the original auction format that Yahoo! has used. On the other hand, when $q_i = r_i$ for all i , bidders are arranged according to the CTR-adjusted bids and bidder $\pi(k)$'s total payment is $c^k r_{\pi(k+1)} b_{\pi(k+1)}$. This corresponds to the original auction format that Google has used. The former is known as the rank-by-bid (RBB) rule, while the latter is known as the rank-by-revenue (RBR) rule.

Following Edelman *et al.* (2007) and Varian (2007), we study the static environment with complete information. One of the reasons is that it is extremely complicated, if not impossible, to analyze the generalized second price auction with multiple positions as a game of incomplete information. Moreover, as these papers claim, the assumption of complete information is a reasonable first approximation since all relevant information about bidders is likely to be inferred over time due to the ease of experimenting with bidding strategies in real-world sponsored search auctions. Varian (2007, p. 1175) notes, ‘It is very easy to experiment with bidding strategies in real-world ad auctions. Google reports click and impression data on an hour-by-hour basis (...) The availability of such tools and services, along with the ease of experimentation, suggest that the full-information assumption is a reasonable first approximation. As we will see below, the Nash equilibrium model seems to fit the observed choices well.’ Edelman *et al.* (2007, p. 249) also notes, ‘... advertisers are likely to learn all relevant information about other’s values.’

Definition 1. A Nash equilibrium is a set of bids $\{b_1, \dots, b_I\}$ that satisfies

$$c^k r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(k+1)} b_{\pi(k+1)}}{q_{\pi(k)}} \right) \geq c^j r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(j+1)} b_{\pi(j+1)}}{q_{\pi(k)}} \right) \text{ for } j > k, \text{ and}$$

$$c^k r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(k+1)} b_{\pi(k+1)}}{q_{\pi(k)}} \right) \geq c^j r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(j)} b_{\pi(j)}}{q_{\pi(k)}} \right) \text{ for } j < k.$$

Hence, bidders do not have incentives to change their assigned positions. Note that this definition reflects the asymmetry: Moving to a higher position requires beating the adjusted bid of who occupies that position, while moving to a lower position requires beating the adjusted bid of who occupies the position next to that position (i.e., the price

the bidder of that position pays). A refinement of the Nash equilibrium concept is proven to be extremely useful: Edelman *et al.* (2007) call it the locally envy-free equilibrium and Varian (2007) calls it the symmetric Nash equilibrium.

Definition 2. A symmetric Nash equilibrium (SNE) is a set of bids $\{b_1, \dots, b_I\}$ that satisfies

$$c^k r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(k+1)} b_{\pi(k+1)}}{q_{\pi(k)}} \right) \geq c^j r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(j+1)} b_{\pi(j+1)}}{q_{\pi(k)}} \right) \quad \text{for } k, j = 1, \dots, I.$$

Equivalently, an SNE set of bids satisfies

$$c^k (q_{\pi(k)} v_{\pi(k)} - q_{\pi(k+1)} b_{\pi(k+1)}) \geq c^j (q_{\pi(k)} v_{\pi(k)} - q_{\pi(j+1)} b_{\pi(j+1)}) \quad \text{for } k, j = 1, \dots, I.$$

Since $q_{\pi(j)} b_{\pi(j)} \geq q_{\pi(j+1)} b_{\pi(j+1)}$, an SNE is a Nash equilibrium. We now list some properties of SNE. The derivations are just a simple extension of those in Varian (2007), which deals with the case when $r_i = 1$ for all i and which does not consider the quality score (or $q_i = 1$ for all i).

Fact 1. Nonnegative surplus: $c^k r_{\pi(k)} (v_{\pi(k)} - q_{\pi(k+1)} b_{\pi(k+1)} / q_{\pi(k)}) \geq 0$.

Proof: When $K < I$, we have

$$c^k r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(k+1)} b_{\pi(k+1)}}{q_{\pi(k)}} \right) \geq c^{K+1} r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(K+2)} b_{\pi(K+2)}}{q_{\pi(k)}} \right) = 0$$

since $c^{K+1} = 0$. When $K = I$, we have

$$c^k r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(k+1)} b_{\pi(k+1)}}{q_{\pi(k)}} \right) \geq c^K r_{\pi(k)} \left(v_{\pi(k)} - \frac{q_{\pi(K+1)} b_{\pi(K+1)}}{q_{\pi(k)}} \right) \geq 0$$

since $b_{\pi(K+1)} = 0$.

Q.E.D.

Fact 2. Monotone values: $q_{\pi(k)} v_{\pi(k)}$ is nonincreasing in $k = 1, \dots, K$, and $q_{\pi(K)} v_{\pi(K)} \geq q_{\pi(j)} v_{\pi(j)}$ for all $j = K + 1, \dots, I$.

Proof: The inequality in Definition 2 can be rearranged as

$$(c^k - c^j) q_{\pi(k)} v_{\pi(k)} \geq c^k q_{\pi(k+1)} b_{\pi(k+1)} - c^j q_{\pi(j+1)} b_{\pi(j+1)}.$$

By exchanging the role of k and j , we also have

$$(c^j - c^k)q_{\pi(j)}v_{\pi(j)} \geq c^j q_{\pi(j+1)}b_{\pi(j+1)} - c^k q_{\pi(k+1)}b_{\pi(k+1)}.$$

Adding the inequalities, we get

$$(c^k - c^j)(q_{\pi(k)}v_{\pi(k)} - q_{\pi(j)}v_{\pi(j)}) \geq 0.$$

The result follows. *Q.E.D.*

Hence, while the bidders are ordered according to adjusted bids, it turns out that adjusted valuations have the same order as the adjusted bids in SNEs. Observe that the sum of advertisers' total payoffs is $\sum_{k=1}^K c^k r_{\pi(k)} v_{\pi(k)}$. This sum is maximized when bidders are placed according to the order of $r_i v_i$'s, that is, when a bidder with the highest $r_i v_i$ is assigned to position 1, and a bidder with the second highest $r_i v_i$ is assigned to position 2, and so on. Then, it is easy to see from Fact 2 that this sum is maximized when $q_i = r_i$ for all i , or more generally, when (q_1, \dots, q_I) are set to respect the order of $r_i v_i$'s.⁷

Fact 3. Non-monotone payments: $c^k q_{\pi(k+1)} b_{\pi(k+1)}$ is nonincreasing in $k = 1, \dots, I$, but the payment $c^k r_{\pi(k)} q_{\pi(k+1)} b_{\pi(k+1)} / q_{\pi(k)}$ may not be monotone.

Proof: Observe that $c^{k-1} q_{\pi(k)} b_{\pi(k)} \geq c^k q_{\pi(k+1)} b_{\pi(k+1)}$ since $c^{k-1} \geq c^k$ and $q_{\pi(k)} b_{\pi(k)} \geq q_{\pi(k+1)} b_{\pi(k+1)}$. However, it is possible to have

$$c^{k-1} r_{\pi(k-1)} \frac{q_{\pi(k)} b_{\pi(k)}}{q_{\pi(k-1)}} < c^k r_{\pi(k)} \frac{q_{\pi(k+1)} b_{\pi(k+1)}}{q_{\pi(k)}}$$

when $r_{\pi(k-1)} / q_{\pi(k-1)}$ is sufficiently smaller than $r_{\pi(k)} / q_{\pi(k)}$. Hence, payments may not be decreasing in k . *Q.E.D.*

Fact 4. One step solution: If a set of bids satisfies the inequality in Definition 2 for two adjacent positions, then it satisfies the inequality for all positions. That is, if the inequality

⁷ Some authors call an outcome that maximizes the sum of advertisers' total payoffs as an 'efficient' outcome. We believe this terminology is somewhat misleading since this outcome does not consider consumers' payoffs.

$c^k(q_{\pi(k)}v_{\pi(k)} - q_{\pi(k+1)}b_{\pi(k+1)}) \geq c^j(q_{\pi(k)}v_{\pi(k)} - q_{\pi(j+1)}b_{\pi(j+1)})$ holds for each $k = 1, \dots, I$ and $j = k - 1$ and $k + 1$, then Definition 2 holds for all $k, j = 1, \dots, I$.⁸

Proof: Fix $k = 1, \dots, I$. First consider the case when $j > k$. By hypothesis, we have inequalities

$$\begin{aligned} c^k(q_{\pi(k)}v_{\pi(k)} - q_{\pi(k+1)}b_{\pi(k+1)}) &\geq c^{k+1}(q_{\pi(k)}v_{\pi(k)} - q_{\pi(k+2)}b_{\pi(k+2)}), \\ c^{k+1}(q_{\pi(k+1)}v_{\pi(k+1)} - q_{\pi(k+2)}b_{\pi(k+2)}) &\geq c^{k+2}(q_{\pi(k+1)}v_{\pi(k+1)} - q_{\pi(k+3)}b_{\pi(k+3)}), \\ &\vdots \\ c^{j-1}(q_{\pi(j-1)}v_{\pi(j-1)} - q_{\pi(j)}b_{\pi(j)}) &\geq c^j(q_{\pi(j-1)}v_{\pi(j-1)} - q_{\pi(j+1)}b_{\pi(j+1)}). \end{aligned}$$

Since $q_{\pi(k)}v_{\pi(k)} \geq q_{\pi(l)}v_{\pi(l)}$ for $l = k + 1, \dots, j - 1$ by Fact 2, as well as $c^1 > c^2 > \dots > c^K > 0 = c^{K+1} = \dots = c^I$, the inequalities still hold even when we replace $q_{\pi(l)}v_{\pi(l)}$'s with $q_{\pi(k)}v_{\pi(k)}$. Canceling out the redundant terms, we get

$$c^k(q_{\pi(k)}v_{\pi(k)} - q_{\pi(k+1)}b_{\pi(k+1)}) \geq c^j(q_{\pi(k)}v_{\pi(k)} - q_{\pi(j+1)}b_{\pi(j+1)}).$$

The case when $j < k$ is similar, using the other adjacent inequalities. *Q.E.D.*

Definition 2 of SNE gives the inequalities

$$\begin{aligned} c^k(q_{\pi(k)}v_{\pi(k)} - q_{\pi(k+1)}b_{\pi(k+1)}) &\geq c^{k+1}(q_{\pi(k)}v_{\pi(k)} - q_{\pi(k+2)}b_{\pi(k+2)}) \quad \text{and} \\ c^{k+1}(q_{\pi(k+1)}v_{\pi(k+1)} - q_{\pi(k+2)}b_{\pi(k+2)}) &\geq c^k(q_{\pi(k+1)}v_{\pi(k+1)} - q_{\pi(k+1)}b_{\pi(k+1)}), \end{aligned}$$

which can be combined to get

$$\begin{aligned} (c^k - c^{k+1})q_{\pi(k+1)}v_{\pi(k+1)} + c^{k+1}q_{\pi(k+2)}b_{\pi(k+2)} &\leq c^kq_{\pi(k+1)}b_{\pi(k+1)} \\ &\leq (c^k - c^{k+1})q_{\pi(k)}v_{\pi(k)} + c^{k+1}q_{\pi(k+2)}b_{\pi(k+2)}. \end{aligned}$$

Thus, each bidder's $c^kq_{\pi(k+1)}b_{\pi(k+1)}$ is bounded below and above. We can express the upper and lower boundary cases in the previous inequalities recursively as

$$\begin{aligned} c^kq_{\pi(k+1)}b_{\pi(k+1)}^U &= (c^k - c^{k+1})q_{\pi(k)}v_{\pi(k)} + c^{k+1}q_{\pi(k+2)}b_{\pi(k+2)}^U \quad \text{and} \\ c^kq_{\pi(k+1)}b_{\pi(k+1)}^L &= (c^k - c^{k+1})q_{\pi(k+1)}v_{\pi(k+1)} + c^{k+1}q_{\pi(k+2)}b_{\pi(k+2)}^L, \end{aligned}$$

⁸ Needless to say, there exists only one relevant adjacent inequality when $k = 1$ or $k = I$.

from which we have

$$c^k q_{\pi(k+1)} b_{\pi(k+1)}^U = \sum_{j=k}^K (c^j - c^{j+1}) q_{\pi(j)} v_{\pi(j)},$$

$$c^k q_{\pi(k+1)} b_{\pi(k+1)}^L = \sum_{j=k}^K (c^j - c^{j+1}) q_{\pi(j+1)} v_{\pi(j+1)}.$$

2.2. Optimal quality scores: full extraction of the surplus

The main question we ask in this paper is how to set the quality scores (q_1, \dots, q_I) to maximize the auctioneer's revenue. That is, we take the quality score as a strategic decision variable. The lower bound of the auctioneer's revenue is given by

$$\begin{aligned} \sum_{k=1}^K \frac{r_{\pi(k)}}{q_{\pi(k)}} c^k q_{\pi(k+1)} b_{\pi(k+1)}^L &= \sum_{k=1}^K \frac{r_{\pi(k)}}{q_{\pi(k)}} \sum_{j=k}^K (c^j - c^{j+1}) q_{\pi(j+1)} v_{\pi(j+1)} \\ &= \sum_{k=1}^K (c^k - c^{k+1}) q_{\pi(k+1)} v_{\pi(k+1)} \left(\sum_{j=1}^k \frac{r_{\pi(j)}}{q_{\pi(j)}} \right) \end{aligned}$$

and the upper bound can be similarly expressed. We will henceforth work with the lower bound. This bound is prominent in that it coincides with the Vickrey payment when the quality score q_i is set to the click-through rate r_i . Edelman *et al.* (2007) give special attention to the lower bound in their Theorem 1. Varian (2007) also argues that the lower bound is the most plausible outcome. Moreover, as we show in Proposition 2 below, the optimal revenue even with the lower bound SNE extracts all the bidders' surplus, hence all the other SNEs *a fortiori* achieve the full extraction of surplus.

From now on, we will let $r_1 v_1 \geq r_2 v_2 \geq \dots \geq r_I v_I$ without loss of generality. That is, we rename the bidders in the order of $r_i v_i$'s. We have:

Proposition 1. *If the quality scores (q_1, \dots, q_I) maximize the lower bound of the auctioneer's revenue, then $q_1 v_1 \geq q_2 v_2 \geq \dots \geq q_K v_K$ and $q_K v_K \geq q_j v_j$ for all $j = K + 1, \dots, I$.*

Proof: Fix the quality scores (q_1, \dots, q_I) , and let $\pi : I \rightarrow I$ be the order of bidders induced by these quality scores. That is, $q_{\pi(1)} v_{\pi(1)} \geq \dots \geq q_{\pi(I)} v_{\pi(I)}$. We prove by contradiction,

so suppose that either (i) there exists $k = 1, \dots, K - 1$ such that $i = \pi(k)$, $j = \pi(k + 1)$, $j < i$ and $q_i v_i > q_j v_j$, or (ii) no such k exists but $q_i v_i > q_j v_j$ for some $j = 1, \dots, K$ and $i = K + 1, \dots, I$.

Consider case (i) first. We will show that we can choose another set of quality scores, which results in an increase of the auctioneer's revenue. With the original quality scores, we have

$$q_{\pi(1)} v_{\pi(1)} \geq \dots \geq q_{\pi(k-1)} v_{\pi(k-1)} \geq q_i v_i > q_j v_j \geq q_{\pi(k+2)} v_{\pi(k+2)} \geq \dots \geq q_{\pi(I)} v_{\pi(I)}. \quad (1)$$

Now construct new quality scores by decreasing $q_{\pi(1)}, \dots, q_{\pi(k-1)}$ and q_i so that

$$q_{\pi(1)}^0 v_{\pi(1)} = q_{\pi(2)}^0 v_{\pi(2)} = \dots = q_{\pi(k-1)}^0 v_{\pi(k-1)} = q_i^0 v_i = q_j v_j, \quad (2)$$

i.e. decrease the quality scores from 1 to k so that the adjusted valuations are equal to $q_j v_j = q_{\pi(k+1)} v_{\pi(k+1)}$, while leaving $q_{\pi(k+1)}, \dots, q_{\pi(I)}$ intact. With the new quality scores, we have

$$q_{\pi(1)}^0 v_{\pi(1)} \geq \dots \geq q_{\pi(k-1)}^0 v_{\pi(k-1)} \geq q_j v_j \geq q_i^0 v_i \geq q_{\pi(k+2)} v_{\pi(k+2)} \geq \dots \geq q_{\pi(I)} v_{\pi(I)}.$$

Note well that we choose the order of adjusted valuations in a way to switch only the positions of i and j , though all the adjusted valuations from 1 to $k + 1$ have the same value of $q_j v_j$ by construction. This is an innocuous convention only to avoid unnecessary complications: We can alternatively choose $q_{\pi(l)}^0 v_{\pi(l)} = q_j v_j + \epsilon/l$ for $l = 1, \dots, k - 1$, $q_j^0 v_j = q_j v_j + \epsilon/k$ and $q_i^0 v_i = q_j v_j$ and derive the same conclusion.

The lower bound of the auctioneer's revenue with the original (q_1, \dots, q_I) is

$$\begin{aligned}
R &= (c^1 - c^2)q_{\pi(2)}v_{\pi(2)}\frac{r_{\pi(1)}}{q_{\pi(1)}} + (c^2 - c^3)q_{\pi(3)}v_{\pi(3)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}} + \frac{r_{\pi(2)}}{q_{\pi(2)}}\right) \\
&+ \dots \\
&+ (c^{k-2} - c^{k-1})q_{\pi(k-1)}v_{\pi(k-1)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}} + \dots + \frac{r_{\pi(k-2)}}{q_{\pi(k-2)}}\right) \\
&+ (c^{k-1} - c^k)q_i v_i \left(\frac{r_{\pi(1)}}{q_{\pi(1)}} + \dots + \frac{r_{\pi(k-1)}}{q_{\pi(k-1)}}\right) \\
&+ (c^k - c^{k+1})q_j v_j \left(\frac{r_{\pi(1)}}{q_{\pi(1)}} + \dots + \frac{r_{\pi(k)}}{q_{\pi(k)}}\right) \\
&+ (c^{k+1} - c^{k+2})q_{\pi(k+2)}v_{\pi(k+2)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}} + \dots + \frac{r_{\pi(k+1)}}{q_{\pi(k+1)}}\right) \\
&+ \dots \\
&+ (c^K - c^{K+1})q_{\pi(K+1)}v_{\pi(K+1)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}} + \dots + \frac{r_{\pi(K)}}{q_{\pi(K)}}\right).
\end{aligned}$$

The auctioneer's revenue with the new quality scores is

$$\begin{aligned}
R^0 &= (c^1 - c^2)q_{\pi(2)}^0 v_{\pi(2)}\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + (c^2 - c^3)q_{\pi(3)}^0 v_{\pi(3)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + \frac{r_{\pi(2)}}{q_{\pi(2)}^0}\right) \\
&+ \dots \\
&+ (c^{k-2} - c^{k-1})q_{\pi(k-1)}^0 v_{\pi(k-1)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + \dots + \frac{r_{\pi(k-2)}}{q_{\pi(k-2)}^0}\right) \\
&+ (c^{k-1} - c^k)q_j v_j \left(\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + \dots + \frac{r_{\pi(k-1)}}{q_{\pi(k-1)}^0}\right) \\
&+ (c^k - c^{k+1})q_i^0 v_i \left(\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + \dots + \frac{r_{\pi(k-1)}}{q_{\pi(k-1)}^0} + \frac{r_j}{q_j}\right) \\
&+ (c^{k+1} - c^{k+2})q_{\pi(k+2)}v_{\pi(k+2)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + \dots + \frac{r_j}{q_j} + \frac{r_i}{q_i}\right) \\
&+ \dots \\
&+ (c^K - c^{K+1})q_{\pi(K+1)}v_{\pi(K+1)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + \dots + \frac{r_j}{q_j} + \frac{r_i}{q_i} + \dots + \frac{r_{\pi(K)}}{q_{\pi(K)}}\right).
\end{aligned}$$

The first $k - 2$ terms of R^0 are not less than the corresponding terms in R since, for $l = 1, \dots, k - 2$,

$$\begin{aligned}
&(c^l - c^{l+1})q_{\pi(l+1)}^0 v_{\pi(l+1)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + \dots + \frac{r_{\pi(l)}}{q_{\pi(l)}^0}\right) \\
&= (c^l - c^{l+1})(r_{\pi(1)}v_{\pi(1)} + \dots + r_{\pi(l)}v_{\pi(l)}) && \text{by (2)} \\
&\geq (c^l - c^{l+1})q_{\pi(l+1)}v_{\pi(l+1)}\left(\frac{r_{\pi(1)}}{q_{\pi(1)}} + \dots + \frac{r_{\pi(l)}}{q_{\pi(l)}}\right). && \text{by (1)}
\end{aligned}$$

Likewise, the $(k-1)$ -th term of R^0 is not less than the corresponding terms in R since

$$\begin{aligned} & (c^{k-1} - c^k)q_j v_j \left(\frac{r_{\pi(1)}}{q_{\pi(1)}^0} + \cdots + \frac{r_{\pi(k-1)}}{q_{\pi(k-1)}^0} \right) \\ &= (c^{k-1} - c^k)(r_{\pi(1)}v_{\pi(1)} + \cdots + r_{\pi(k-1)}v_{\pi(k-1)}) \\ &\geq (c^{k-1} - c^k)q_i v_i \left(\frac{r_{\pi(1)}}{q_{\pi(1)}} + \cdots + \frac{r_{\pi(k-1)}}{q_{\pi(k-1)}} \right). \end{aligned}$$

Hence,

$$\begin{aligned} R^0 - R &\geq (c^k - c^{k+1})q_j v_j \left(\sum_{l=1}^{k-1} \frac{r_{\pi(l)}}{q_{\pi(l)}^0} - \sum_{l=1}^{k-1} \frac{r_{\pi(l)}}{q_{\pi(l)}} + \frac{r_j}{q_j} - \frac{r_i}{q_i} \right) \\ &\quad + (c^{k+1} - c^{k+2})q_{\pi(k+2)}v_{\pi(k+2)} \left(\sum_{l=1}^{k-1} \frac{r_{\pi(l)}}{q_{\pi(l)}^0} - \sum_{l=1}^{k-1} \frac{r_{\pi(l)}}{q_{\pi(l)}} + \frac{r_i}{q_i^0} - \frac{r_i}{q_i} \right) \\ &\quad + \cdots \\ &\quad + (c^K - c^{K+1})q_{\pi(K+1)}v_{\pi(K+1)} \left(\sum_{l=1}^{k-1} \frac{r_{\pi(l)}}{q_{\pi(l)}^0} - \sum_{l=1}^{k-1} \frac{r_{\pi(l)}}{q_{\pi(l)}} + \frac{r_i}{q_i^0} - \frac{r_i}{q_i} \right) \\ &> 0. \end{aligned}$$

since (i) $q_{\pi(l)}^0 < q_{\pi(l)}$ for $l = 1, \dots, k-1$ as well as $q_i^0 < q_i$, and (ii) $\frac{r_j}{q_j} > \frac{r_i}{q_i}$ which follows from the facts that $r_j v_j \geq r_i v_i$ but $q_i v_i > q_j v_j$.

Consider case (ii) next. There must exist $j = \pi(k)$ with $j \leq K < k$, and $q_i v_i > q_j v_j$ for $i > K$. We construct new quality scores by increasing q_j so that $q_j^0 v_j = q_{\pi(K)} v_{\pi(K)}$, while leaving other quality scores intact. In these new quality scores, the only change is that bidder j has moved up to the K -th position while pushing other bidders in $\{\pi(K), \dots, \pi(I)\}$ down by one rank. Hence, the change in the auctioneer's revenue $R^0 - R$ is⁹

$$\begin{aligned} & (c^K - c^{K+1}) \left(q_{\pi(K)} v_{\pi(K)} \left(\sum_{l=1}^{K-1} \frac{r_{\pi(l)}}{q_{\pi(l)}} + \frac{r_j}{q_j^0} \right) - q_{\pi(K+1)} v_{\pi(K+1)} \left(\sum_{l=1}^{K-1} \frac{r_{\pi(l)}}{q_{\pi(l)}} + \frac{r_{\pi(K)}}{q_{\pi(K)}} \right) \right) \\ &\geq (c^K - c^{K+1})(r_j v_j - r_{\pi(K)} v_{\pi(K)}) > 0 \end{aligned}$$

since it is safe to assume that $r_j v_j > r_{\pi(K)} v_{\pi(K)}$. (Otherwise, i.e., if $r_j v_j = r_{\pi(K)} v_{\pi(K)}$, then we are back to case (i) for the new quality scores, possibly after the recursive switching

⁹ Note that the first inequality follows from replacing $q_{\pi(K+1)} v_{\pi(K+1)}$ with $q_{\pi(K)} v_{\pi(K)}$.

of adjacent positions starting from j back to i if equalities of adjusted valuations remain.)
Q.E.D.

The following example illustrates the algorithm in the proof.

Example 1. There are three positions with $c^1 = 3$, $c^2 = 2$, and $c^3 = 1$, and three bidders with $v_1 = 3, v_2 = 2, v_3 = 1$, and $r_1 = r_2 = r_3 = 1$. Suppose that $q_2 v_2 > q_3 v_3 > q_1 v_1$, say $q_1 = 1, q_2 = 3, q_3 = 4$. Hence, $\pi(1) = 2, \pi(2) = 3$, and $\pi(3) = 1$. The auctioneer's revenue is

$$R = (c^1 - c^2)q_3 v_3 \frac{1}{q_2} + (c^2 - c^3)q_1 v_1 \left(\frac{1}{q_2} + \frac{1}{q_3}\right) = \frac{4}{3} + 3\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{37}{12}.$$

Now decrease q_2 and q_3 so that $q_2^0 = \frac{3}{2}$ and $q_3^0 = 3$. This gives $q_2^0 v_2 \geq q_1 v_1 \geq q_3^0 v_3$. The auctioneer's revenue becomes

$$R^0 = (c^1 - c^2)q_1 v_1 \frac{1}{q_2^0} + (c^2 - c^3)q_3^0 v_3 \left(\frac{1}{q_2^0} + \frac{1}{q_1}\right) = 2 + 3\left(\frac{2}{3} + 1\right) = 7.$$

Proposition 1 shows that an outcome that maximizes the auctioneer's revenue respects the order of the CTR-adjusted valuations $r_i v_i$'s. So, in a revenue-maximizing outcome, the bidder with the highest CTR-adjusted valuation is assigned to position 1, the bidder with the second highest CTR-adjusted valuation is assigned to position 2, and so on.¹⁰ Recall that this outcome also maximizes the sum of advertisers' total payoffs as we saw after Fact 2.

We want to note that the auctioneer's revenue in reality depends not only on the CTR but also on many other factors including the relevance of the keyword and the landing page quality. Since the current paper abstracts away from real-world complications and focuses only on the CTR as the revenue-relevant variable, Proposition 1 should be understood in a way that the revenue-maximizing outcome respects the order of the adjusted valuations that incorporate all revenue-relevant variables.

¹⁰ To put it precisely, let $\phi : I \rightarrow I$ be the permutation of bidders so that $\phi(i)$ is the bidder with the i -th highest CTR-adjusted valuation. That is, $r_{\phi(1)} v_{\phi(1)} \geq \dots \geq r_{\phi(I)} v_{\phi(I)}$. Proposition 1 shows that the assignment order $\pi : I \rightarrow I$ induced by (q_1, \dots, q_I) is identical to ϕ in a revenue-maximizing outcome, up to revenue-irrelevant equivalent classes.

We next show that the auctioneer can, in fact, extract all the advertisers' surplus by setting quality scores optimally when the number of bidders exceeds the number of positions.

Proposition 2. *(Full extraction of advertisers' surplus) Assume $K < I$. The optimal quality scores (q_1^*, \dots, q_I^*) that maximize the lower bound of the auctioneer's revenue satisfy*

$$q_1^* v_1 = q_2^* v_2 = \dots = q_{K+1}^* v_{K+1} \geq q_j^* v_j \quad \text{for all } j = K + 2, \dots, I$$

and the auctioneer's optimal revenue is equal to the maximal sum of advertisers' total payoffs, $\sum_{k=1}^K c^k r_k v_k$.¹¹

Proof: The revenue from position K is $\frac{r_K}{q_K}(c^K - c^{K+1})q_{K+1}v_{K+1}$. Given q_K , it is optimal to raise q_{K+1} as high as possible. By the restriction $q_K v_K \geq q_{K+1} v_{K+1}$ from Proposition 1, we have $q_{K+1} = q_K v_K / v_{K+1}$ and the corresponding revenue is $c^K r_K v_K$ since $c^{K+1} = 0$. By backward induction, the revenue from position $k = 1, \dots, K$ is

$$\frac{r_k}{q_k} \left(\sum_{j=k}^K (c^j - c^{j+1}) q_{j+1} v_{j+1} \right) = \frac{r_k}{q_k} (c^k - c^{K+1}) q_{k+1} v_{k+1}.$$

Given q_k , it is optimal to choose $q_{k+1} = q_k v_k / v_{k+1}$ by Proposition 1. The resulting revenue for position k is $c^k r_k v_k$. On the other hand, the quality scores q_j^* for $j = K + 2, \dots, I$ need only to satisfy the restrictions given by Proposition 1 since bidders without positions do not pay. *Q.E.D.*

Hence, the auctioneer can extract all the advertisers' surplus in the lower bound SNE. Since bidders enjoy a nonnegative surplus in any SNE by Fact 1, this proposition implies that this is also the best outcome for the auctioneer in any SNE.

¹¹ One obvious choice of the quality scores is to set $q_1^* = 1$, $q_k^* = v_1 / v_k$ for $k = 2, \dots, K + 1$, and $q_j^* \leq v_1 / v_j$ for $j = K + 2, \dots, I$.

What is the reason behind this result? As for the equilibrium bids and payments, we have:

Corollary 3. *Assume $K < I$. With the optimal quality scores (q_1^*, \dots, q_I^*) , we have $b_k = v_k$ for $k = 2, \dots, K+1$, and the payment per click for bidder k is v_k for $k = 1, \dots, K$.*

Proof: Easy to obtain from Proposition 2 and Definition 2.

Q.E.D.

The optimal quality scores effectively set all the bidders' valuations equal to the highest valuation, which induces intense bidding competition. In the optimal auction design under incomplete information, Myerson (1981) has shown that the auctioneer can increase revenue by giving bid preferences to weak bidders whose expected willingness to pay is lower.¹² Observe that optimal quality scores work similarly as bid preferences in our complete information setting of the generalized second-price auction. Observe also that optimal quality scores not only increase the auctioneer's revenue but in fact extract all the advertisers' surplus.¹³

Notwithstanding the full extraction result, it should be noted that the short-term incentive of exploitation may be checked by the long-term incentive of market cultivation. Simply put, sponsored search auctions cannot survive in the end unless advertisers find their surplus satisfactory. Hence, the advertising intermediaries such as Google and Yahoo! may wish to guarantee normal profits to the advertisers. The following proposition shows that it is possible to extract any portion of advertisers' surplus.

Proposition 4. *Assume $K < I$. For any $\alpha \in [0, 1]$, the auctioneer can choose the quality scores so that, for all $k = 1, \dots, K$, the revenue from position k is exactly 100α percent of the total advertiser's surplus $c^k r_k v_k$ who occupies that position.*

¹² Liu and Chen (2006) exploit this insight in the context of sponsored search auctions. They consider the first-price auction format and only the case of a single position, which are at wide variance with actual practice. See other related papers in their reference, too. On the other hand, Lahaie and Pennock (2007) start discussing quality scores under complete information setting. They move on to the incomplete information setting and, with a specific form of $q_i = r_i^\alpha$ for the quality score, show by simulation that higher correlation between r_i and v_i leads to a smaller optimal α .

¹³ We conjecture that full extraction may not be feasible under incomplete information unless valuations are correlated across advertisers in the spirit of Crémer and McLean (1988).

Proof: The revenue from position K is $\frac{r_K}{q_K}(c^K - c^{K+1})q_{K+1}v_{K+1}$. By choosing $q_{K+1} = \frac{\alpha q_K v_K}{v_{K+1}}$, that is, by setting $q_{K+1}v_{K+1} = \alpha q_K v_K$, it becomes $\alpha c^K r_K v_K$. Next, the revenue from position $K - 1$ is

$$\begin{aligned} & \frac{r_{K-1}}{q_{K-1}} [(c^{K-1} - c^K)q_k v_K + (c^K - c^{K+1})q_{K+1}v_{K+1}] \\ &= \frac{r_{K-1}}{q_{K-1}} [(c^{K-1} - c^K)q_k v_K + \alpha(c^K - c^{K+1})]q_K v_K = \frac{r_{K-1}}{q_{K-1}} [c^{K-1} - (1 - \alpha)c^K]q_k v_K. \end{aligned}$$

By choosing $q_K = \frac{\alpha c^{K-1} q_{K-1} v_{K-1}}{[c^{K-1} - (1 - \alpha)c^K]v_K}$, that is, by setting $[c^{K-1} - (1 - \alpha)c^K]q_K v_K = \alpha c^{K-1} q_{K-1} v_{K-1}$, it becomes $\alpha c^{K-1} r_{K-1} v_{K-1}$. By backward induction, the revenue from position $k = 1, \dots, K$ becomes $\alpha c^k r_k v_k$ by choosing

$$q_{k+1} = \frac{\alpha c^k q_k v_k}{[c^k - (1 - \alpha)c^{k+1}]v_{k+1}},$$

that is, by setting $[c^k - (1 - \alpha)c^{k+1}]q_{k+1}v_{k+1} = \alpha c^k q_k v_k$. Note well that these quality scores preserve the order of CTR-adjusted valuations so that bidder k occupies position k for all $k = 1, \dots, K$. *Q.E.D.*

We finally mention that full extraction does not occur when $K = I$ since there cannot be enough competition for positions. A straightforward derivation along the lines of the proof of Proposition 2 shows the following: The revenue from position K is zero, and generally the revenue from position k is $(c^k - c^K)r_k v_k$. Bidder k 's surplus is $c^K r_k v_k = c^k r_k v_k - (c^k - c^K)r_k v_k > 0$ for $k = 1, \dots, K$.

3 Discussion

We have shown that the quality scores can be optimally chosen to extract all (or any portion) of the advertisers' surplus. This proposition was established in the static model of complete information. As we have discussed before Definition 1, this modeling choice is a reasonable first approximation since all relevant information about advertisers are likely to be learned via frequent interactions and experimentations. Nevertheless, the complete information assumption may be problematic if it is hard to identify which of its equilibrium

predictions are meaningful approximations. Fortunately in this regard, we have established that all equilibria of the complete information game leads to the same conclusion of full surplus extraction.

One may argue that the full extraction result is not surprising or even trivial since, under complete information setting, the advertising intermediary can charge each advertiser's true valuation. Observe however that the advertising intermediary is prevented from directly charging true valuations, nor bidders pay their own bids, under the trading institution of the generalized second price auction. To put it differently, suppose an advertising intermediary gets to know advertisers' true valuations, but it cannot directly charge these valuations due to the trading rule established in the business. What we have shown is that the quality score may be used as an effective instrument to charge true valuations in an auction that appears to set lower prices than bidders' stated valuations.

REFERENCES

- Aggarwal, G., Goel, A., Motwani, R. (2006), "Truthful auctions for pricing search keywords," Proceedings of the 2006 ACM Conference on Electronic Commerce.
- Athey, S., Ellison, G. (2007), "Position auctions with consumer search," Manuscript.
- Battelle, J. (2005), *The Search*, Portfolio Hardcover.
- Börger, T., Cox, I., Pesendorfer, M., Petricek, V. (2007), "Equilibrium bids in auctions of sponsored links: Theory and evidence," Manuscript.
- Crémer, J., McLean, R. (1988), "Full extraction of the surplus in Bayesian and dominant strategy auctions," *Econometrica* 56, 1247-1257.
- Edelman, B., Ostrovsky, M., Schwarz, M. (2007), "Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords," *American Economic Review* 97, 242-259.
- Evans, D. (2008), "The economics of the online advertising industry," *Review of Network Economics* 7, Article 2.

- Evans, D. (2009), "The online advertising industry: Economics, evolution, and privacy," *Journal of Economic Perspective* 23, 37-60.
- Lahaie, S., Pennock, D. (2007), "Revenue analysis of a family of ranking rules for keyword auctions," Proceedings of the 2007 ACM Conference on Electronic Commerce.
- Liu, D., Chen, J. (2006), "Designing online auctions with past performance information," *Decision Support Systems* 42, 1307-1320.
- Liu, D., Chen, J., Whinston, B. (2009), "Current issues in keyword auctions," In Adomavicius, G. and Gupta, A. (eds.), *Business Computing (Handbooks in Information Systems, Vol. 3)*, 69-97.
- Milgrom, P. (2009), "Simplified mechanisms with an application to sponsored-search auctions," *Games and Economic Behavior*, Forthcoming.
- Myerson, R. (1981), "Optimal auction design," *Mathematics of Operations Research* 6, 58-73.
- Varian, H. (2007), "Position auctions," *International Journal of Industrial Organization* 25, 1163-1178.