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The participatory Vickrey–Clarke–Groves mechanism

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Abstract

This paper introduces the participatory Vickrey–Clarke–Groves mechanism, which satisfies both ex-ante budget balance and interim individual rationality. We bound the efficiency loss of this mechanism by a parameter that captures the structure of marginal contributions to the social welfare. We then apply the theory to quite general multiple unit double auction problems to show that the participatory VCG mechanism achieves asymptotic efficiency.
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1. Introduction

The Vickrey–Clarke–Groves (VCG) mechanisms hold enormous theoretical value in the mechanism design theory. They are dominant strategy incentive compatible and outcome efficient.[1] In fact, every dominant strategy incentive compatible and efficient mechanism is a VCG mechanism with a properly chosen transfer rule. Moreover, by the equivalence result of Mookherjee and Reichelstein (1992); Williams (1999) and Krishna and Perry (2000), every Bayesian incentive compatible and efficient mechanism is payoff-equivalent to some VCG mechanism from an interim perspective. Therefore, we can restrict our attention only to the family of VCG mechanisms when we search for a mechanism with an efficient Bayesian–Nash equilibrium that also satisfies other constraints, as long as these other constraints are ex-ante or interim in nature.[2]

The VCG mechanism, however, does not in general satisfy both ex-ante budget balance and interim individual rationality. This is unsatisfactory since these two constraints are quite natural requirements for any mechanism that is designed for the individuals who voluntarily decide whether to participate.

In this paper, we modify the VCG mechanism in a way that the new mechanism always satisfies ex-ante budget balance and interim individual rationality. In particular, we introduce the participation stage in which the mechanism may impose participation fees, which are used to make up for the budget deficit in the later stage of allocation determination. Since individuals decide whether to pay the fee for the right to participate in the mechanism, it is obviously interim individually rational. We call this mechanism *the participatory VCG mechanism.*

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[1] Note that this statement is only true within the private value environment, that which this paper is concerned with.

[2] The recent development in the mechanism design with interdependent valuations also underscores the importance of the VCG mechanisms. See, for example, Ausubel (1999) or Perry and Reny (2002).

The participatory VCG mechanism may not be outcome efficient. The main contribution of this paper is, under a certain submodularity condition which intuitively states that an individual's contribution to the social welfare is decreasing in the size of society, to provide a specific bound on the efficiency loss of the participatory VCG mechanism. This is shown for very general environments (Theorem 2). This bound can be used to prove asymptotic efficiency (and to provide a rate of convergence to full efficiency) for a sequence of environments.

As an illustrative example, we apply the theory to a multiple unit double auction problem. In this setting, there exist many sellers and buyers such that sellers have privately known supply schedules and buyers have privately known demand schedules for multiple units of a good. We show, for a large class of multiple unit double auction problems admitting asymmetry of traders and under a weak condition on the probability measures of traders' valuations, that the participatory VCG mechanism achieves asymptotic efficiency (Theorem 3). The rate of convergence is determined by the probability measures. Compared to the recent surge of interest on multiple unit auctions,[3] virtually no paper has ever characterized the properties of multiple unit double auctions.[4] This paper, therefore, also contributes towards the development of double auction literature comparable to auction literature.

The organization of the paper is as follows. We study the properties of the participatory VCG mechanism in Section 2, and then apply the general theory to a multiple unit double auction problem in Section 3. Some concluding comments are contained in Section 4.

2. The participatory VCG mechanism

2.1. The environment

There is a set $I = \{1, \ldots, n\}$ of individuals. Individual i's private information is represented by a type θ_i. We assume that the set Θ_i of possible types for i is a probability space with measure F_i. We use the usual notation such as $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$, $\Theta = \times_{i=1}^n \Theta_i$, $\Theta_{-i} = \times_{j \neq i} \Theta_j$, and $F = \times_{i=1}^n F_i$. For any nonempty subset J of I, let Y_J denote the set of all possible outcomes or social alternatives achievable among J. We assume that Y_J is independent of $\theta \in \Theta$. The payoff to individual i of type θ_i when i is a member of the set J of participants is $u_i(y; \theta_i) - z_i$, where y is the chosen outcome in Y_J and z_i is the monetary transfer made by i. Therefore, we assume private-value, quasi-linear preferences. We denote the environment by $\mathcal{E} \equiv (I, \{\Theta_i\}_{i \in I}, \{F_i\}_{i \in I}, \{Y_J\}_{J \subseteq I}, \{u_i\}_{i \in I})$.

A direct mechanism is a pair (a, t) where $a : \Theta \to \cup_{J \subseteq I} Y_J$ is an *allocation rule* and $t : \Theta \to \mathbb{R}^n$ is a *transfer rule*. Thus, if reported types are $\theta' \in \Theta$, then $a(\theta')$ is the chosen outcome and $t_i(\theta')$ is the monetary transfer made by individual i. A mechanism (a, t) is dominant strategy incentive compatible (Bayesian incentive compatible, respectively) if truth-telling is a dominant strategy (constitutes a Bayesian–Nash equilibrium, respectively). By the revelation principle, it is with no loss of generality to restrict our attention to direct mechanisms.

An allocation rule $a(\cdot)$ of a direct mechanism is *outcome efficient for J* if

$$a(\theta) \in \text{argmax}_{y \in Y_J} \sum_{i \in J} u_i(y; \theta_i)$$

for all $\theta \in \Theta$. Note that $a(\theta)$ depends only on θ_J, the type profile for J. We will denote an efficient allocation rule for J by $\alpha_J(\cdot)$. It is convenient to define the social welfare for any nonempty subset J of I as

$$w_J(\theta) \equiv \sum_{i \in J} u_i(\alpha_J(\theta); \theta_i).$$

That is, $w_J(\theta)$ is the maximum social welfare achievable among $J \subseteq I$. Note that $w_J(\theta)$ depends only on θ_J. When J is a singleton set such that $J = \{i\}$, the social welfare $w_J(\theta)$ will be called the autarky payoff level and denoted

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[3] For a relatively recent overview of this fast-growing research area, see Klemperer (1999) among others.

[4] As far as the author is aware, this is the first theoretical work that studies efficiency of multiple unit double auctions. A partial list of papers on single unit double auctions (that is, unit supply-unit demand double auctions) are Myerson and Satterthwaite (1983); Wilson (1985); Hagerty and Rogerson (1987); Gresik and Satterthwaite (1989); McAfee (1992); Rustichini et al. (1994); Yoon (2001), and Satterthwaite and Williams (2002). Recently, Cripps and Swinkels (2003) study efficiency of large double auctions.

$w_i(\theta_i)$ to stress individual rationality. That is, $w_i(\theta_i)$ is the maximum payoff level an individual can achieve in isolation.

2.2. The VCG mechanism

A Vickrey–Clarke–Groves mechanism is a direct mechanism for I with the following properties:

(i) The allocation rule is outcome efficient, that is, $a(\cdot) = \alpha_I(\cdot)$, and
(ii) For all $i \in I$ and for all $\theta \in \Theta$, the payment rule is given as

$$\tau_i(\theta) \equiv w_{I \setminus \{i\}}(\theta) - [w_I(\theta) - u_I(\alpha_I(\theta); \theta_i)] + \phi_i(\theta_{-i}).$$

Individual i's payoff in a VCG mechanism is $u_i(\alpha_I(\theta); \theta_i) - \tau_i(\theta) = w_I(\theta) - w_{I \setminus \{i\}}(\theta) - \phi_i(\theta_{-i})$. Define

$$mp_i(\theta) \equiv w_I(\theta) - w_{I \setminus \{i\}}(\theta).$$

Hence, ignoring the term $\phi_i(\theta_{-i})$ which is lump-sum in the sense that it is independent of i's private information θ_i, the VCG mechanism rewards individuals according to their marginal contributions to the social welfare. A VCG mechanism is *interim individually rational* (interim IR henceforth) if and only if

$$E_{\theta_{-i}}[mp_i(\theta)] \geq w_i(\theta_i) + \phi_i$$

for all i and θ_i, where $\phi_i = E_{\theta_{-i}}[\phi_i(\theta_{-i})]$.

The budget deficit of a VCG mechanism (or the subsidy made by the mechanism to individuals) is

$$d(\theta) \equiv -\sum_{i=1}^{n} \tau_i(\theta).$$

Since $d(\theta) = -\sum_{i=1}^{n} \tau_i(\theta) = \sum_{i=1}^{n} [w_I(\theta) - u_i(\alpha_I(\theta); \theta_i) - w_{I \setminus \{i\}}(\theta) - \phi_i(\theta_{-i})] = \sum_{i=1}^{n} [w_I(\theta) - w_{I \setminus \{i\}}(\theta) - \phi_i(\theta_{-i})] - w_I(\theta) = \sum_{i=1}^{n} [mp_i(\theta) - \phi_i(\theta_{-i})] - w_I(\theta)$, a VCG mechanism is *ex-ante budget balancing* (ex-ante BB henceforth) if and only if

$$E_\theta \left[\sum_{i=1}^{n} mp_i(\theta) - w_I(\theta) \right] \leq \sum_{i=1}^{n} \phi_i.$$

It is well known that a VCG mechanism is dominant strategy incentive compatible (DIC henceforth) as well as outcome efficient (EF henceforth).[5] It is also well known that it does not generally satisfy both ex-ante budget balance and interim individual rationality.[6]

Indeed, ex-ante budget balance and interim individual rationality are probably the weakest requirements any plausible mechanism must satisfy. Let us say that a mechanism is *feasible* when it satisfies these two constraints. We now modify the VCG mechanism into a feasible mechanism, namely *the participatory VCG mechanism*. We then provide a specific bound for the efficiency loss of this new mechanism. We also show that the participatory VCG mechanism achieves asymptotic efficiency for many interesting environments.

2.3. The participatory VCG mechanism

For $\phi \equiv (\phi_1, \ldots, \phi_n) \in \mathbb{R}_+^n$, the participatory VCG mechanism with a participation fee structure ϕ, denoted as PVCG(ϕ) henceforth, is defined as follows.

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[5] See Vickrey (1961); Clarke (1971), and Groves (1973).
[6] See Mas-Colell et al. (1995).

Definition 1. PVCG(ϕ)

(i) For each $i \in I$, a lump-sum participation fee ϕ_i is charged by the mechanism for the right to participate.

(ii) For any given set $J \subseteq I$ of participants, outcome efficiency for J is satisfied. That is,

$$a(\theta) \in \mathrm{argmax}_{y \in Y_J} \sum_{i \in J} u_i(y; \theta_i), \quad \text{i.e.,} \quad a(\theta) = \alpha_J(\theta) \text{ for all } \theta \in \Theta.$$

(iii) For any given set $J \subseteq I$ of participants, and for all $i \in J$ and for all $\theta \in \Theta$,

$$\tau_i^J(\theta) \equiv w_{J \setminus \{i\}}(\theta) - [w_J(\theta) - u_i(\alpha_J(\theta); \theta_i)].$$

It is easy to see that PVCG(ϕ) is DIC for any set J of participants as well as outcome efficient for J. In addition, since individuals are free to decide whether to participate, it is also interim IR. That is, individuals will participate in the mechanism if and only if the expected payoff from participation is not less than the autarky payoff level. Participant i's payoff in PVCG(ϕ) when the set of participants is J is given by $u_i(\alpha_J(\theta); \theta_i) - \tau_i^J(\theta) - \phi_i = w_J(\theta) - w_{J \setminus \{i\}}(\theta) - \phi_i$. If we define

$$mp_i^J(\theta) \equiv w_J(\theta) - w_{J \setminus \{i\}}(\theta)$$

then participant i's expected payoff is $E_{\theta_{-i}}[mp_i^J(\theta)] - \phi_i$.[7] Individual i's payoff when he/she does not participate is obviously $w_i(\theta_i)$.

We observe that the social welfare $w_J(\theta)$ satisfies the monotonicity in the sense that, for all $J' \subseteq J \subseteq I$ and $\theta \in \Theta$, the inequality

$$w_J(\theta) \geq w_{J'}(\theta) + \sum_{i \in J \setminus J'} w_i(\theta_i) \tag{*}$$

holds. This is so since, when J is the set of participants, it is an alternative for the mechanism to choose an efficient allocation for J' and let the individuals in $J \setminus J'$ enjoy their autarky payoff levels. This in particular implies that

$$mp_i^J(\theta) \geq w_i(\theta_i) \text{ for all } i, J, \text{ and } \theta.$$

2.4. The efficiency loss

We now determine the set Θ^ϕ of participating types for any given ϕ. Let

$$\mathcal{T} \equiv \{\Theta' \subseteq \Theta | \Theta' = \Theta_1' \times \cdots \times \Theta_n' \quad \text{with } \Theta_i' \subseteq \Theta_i \text{ for all } i \in I\},$$

i.e., $\mathcal{T} \equiv \times_{i=1}^n 2^{\Theta_i}$. For $\Theta' \in \mathcal{T}$, define

$$P_{-i}(\theta; \Theta') \equiv \{j \in I \setminus \{i\} | \theta_j \in \Theta_j'\}$$

and let

$$MP_i(\theta_i; \Theta') \equiv E_{\theta_{-i}}[w_{P_{-i}(\theta; \Theta') \cup \{i\}}(\theta) - w_{P_{-i}(\theta; \Theta')}(\theta)].$$

We define the function $\beta_i^\phi : \mathcal{T} \to 2^{\Theta_i}$ as

$$\beta_i^\phi(\Theta') \equiv \{\theta_i \in \Theta_i | MP_i(\theta_i; \Theta') \geq w_i(\theta_i) + \phi_i\}.$$

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[7] Note that $mp_i(\theta)$ defined in the previous subsection is in fact $mp_i^I(\theta)$ when $J = I$.

Observe that $\beta_i^\phi(\Theta')$ is the set of participating types of i when he/she expects that each individual $j \in I \setminus \{i\}$ will participate if and only if $\theta_j \in \Theta_j'$.[8] Define $\beta^\phi : \mathcal{T} \to \mathcal{T}$ as

$$\beta^\phi(\Theta') \equiv \mathop{\times}_{i=1}^{n} \beta_i^\phi(\Theta').$$

Note that the set Θ^ϕ of participating types is a fixed point of $\beta^\phi(\cdot)$, i.e., $\Theta^\phi = \beta^\phi(\Theta^\phi)$.

It is quite plausible that $\beta^\phi(\cdot)$ has a fixed point in most environments. Nonetheless, we cannot obtain a general fixed point theorem with the current state of knowledge. We introduce a condition.

Condition 1. *There exists $C \subseteq I$ such that $w_{J' \cup \{i\}}(\theta) - w_{J'}(\theta) \geq w_{J \cup \{i\}}(\theta) - w_J(\theta)$ for all J, J' with $C \subseteq J' \subseteq J \subseteq I$, for all $i \in I \setminus J$, and for all $\theta \in \Theta$.*

Condition 1 asserts that the marginal contribution of each individual is non-increasing with respect to the set of participants, given that individuals in C are already present. In other words, each individual's marginal impact on social welfare becomes smaller as the society gets larger. Hence, individuals are substitutes in the sense that each individual's marginal contribution to the social welfare is lower when he joins a larger society. This condition is easily satisfied for many problems, including the multiple unit double auction problem of the next section. We provide a simple example.

Example 1. There are three individuals, that is, $I = \{1, 2, 3\}$. Individual 3 is a seller who initially owns one indivisible unit of a good, while individuals 1 and 2 are potential buyers. Each individual's type θ_i is independently drawn from the uniform distribution on the unit interval $[0, 1]$. Individual i's payoff from the outcome, $u_i(y; \theta_i)$, is equal to θ_i when she finally owns the good, and zero otherwise. Hence, θ_i is individual i's valuation for the good. In an efficient allocation, the good is allocated to an individual with the highest valuation.

Let $C = \{3\}$, and suppose $J' = \{3\}$ and $J = \{2, 3\}$. We have $w_{J' \cup \{1\}}(\theta) - w_{J'}(\theta) = \theta_1 - \theta_3$ if $\theta_1 > \theta_3$, and zero otherwise. We also have $w_{J \cup \{1\}}(\theta) - w_J(\theta) = \theta_1 - \max\{\theta_2, \theta_3\}$ if $\theta_1 > \max\{\theta_2, \theta_3\}$, and zero otherwise. Since $\max\{\theta_2, \theta_3\} \geq \theta_3$ and so $\theta_1 \leq \theta_3$ implies $\theta_1 \leq \max\{\theta_2, \theta_3\}$, we have $w_{J' \cup \{1\}}(\theta) - w_{J'}(\theta) \geq w_{J \cup \{1\}}(\theta) - w_J(\theta)$. The case when $J' = \{3\}$ and $J = \{1, 3\}$ is similar.

We want to note that, if we treat the seller as a non-strategic player who only supplies the good (or equivalently, θ_3 is known to be zero) in the previous example, then this problem becomes an auction problem. We can essentially set $C = \emptyset$ in this case as treating the non-strategic seller implicitly. Ausubel and Milgrom (2002) call Condition 1 the *buyer-submodularity* condition in the package auction context.[9] We note in addition that the inequality in Condition 1 can be reversed for Theorem 1 below. That is, we can also find a fixed point when $w_{J' \cup \{i\}}(\theta) - w_{J'}(\theta) \leq w_{J \cup \{i\}}(\theta) - w_J(\theta)$ holds for all J, J' with $C \subseteq J' \subseteq J \subseteq I$, for all $i \in I \setminus J$, and for all $\theta \in \Theta$. Observe that this condition is the "convexity" condition introduced in Shapley (1971), which states that individuals are complements rather than substitutes. We finally note that Condition 1 implies the following condition.

Condition 2. *There exists $C \subseteq I$ such that $w_I(\theta) - w_J(\theta) \leq \sum_{i \in I \setminus J}[w_{J \cup \{i\}}(\theta) - w_J(\theta)]$ for all J with $C \subseteq J \subseteq I$ and for all $\theta \in \Theta$.*

To see this, fix J with $C \subseteq J \subseteq I$ and $\theta \in \Theta$, and name individuals in $I \setminus J$ as $i_1, i_2, \ldots, i_K$. Then, $\sum_{i \in I \setminus J}[w_{J \cup \{i\}}(\theta) - w_J(\theta)] = \sum_{k=1}^{K}[w_{J \cup \{i_k\}}(\theta) - w_J(\theta)] \geq [w_{J \cup \{i_1\}}(\theta) - w_J(\theta)] + [w_{J \cup \{i_1, i_2\}}(\theta) - w_{J \cup \{i_1\}}(\theta)] + [w_{J \cup \{i_1, i_2, i_3\}}(\theta) - w_{J \cup \{i_1, i_2\}}(\theta)] + \cdots + [w_I(\theta) - w_{I \setminus \{i_K\}}(\theta)] = w_I(\theta) - w_J(\theta)$, where the inequality follows from Condition 1.

The term $w_{J \cup \{i\}}(\theta) - w_J(\theta)$ is the marginal contribution of individual i, and the term $w_I(\theta) - w_J(\theta)$ is the marginal contribution of $I \setminus J$ as a group to the social welfare when individuals in J are already present. Therefore, Condition 2 imposes: there exists a critical mass C of individuals such that, whenever a group J of individuals including C are

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[8] We assume that individuals will participate when they are indifferent between participation and non-participation.

[9] It is interesting that the same condition is needed for different results. See Theorems 7 and 8 of Ausubel and Milgrom (2002) for their results. They also show in their Theorem 11 that this condition is satisfied when goods are substitutes.

already present, the sum of each remaining individual's marginal contribution is at least as large as the total marginal contribution of the remaining individuals as a group.

We now present the following theorem.

Theorem 1. *The function $\beta^\phi : \mathcal{T} \to \mathcal{T}$ for a given ϕ has a fixed point if $\beta^\phi(\cdot)$ is monotonic. In particular, if the underlying environment satisfies* Condition 1, *then we can find ϕ with $\phi_i = 0$ for all $i \in C$ such that the corresponding $\beta^\phi(\cdot)$ has a fixed point.*

Proof. If $\beta^\phi(\Theta') \subseteq \beta^\phi(\Theta'')$ for all $\Theta' \subseteq \Theta''$, define a partial order $\le$ on $\mathcal{T}$ as $\Theta' \le \Theta''$ iff $\Theta' \subseteq \Theta''$, and if $\beta^\phi(\Theta') \supseteq \beta^\phi(\Theta'')$ for all $\Theta' \subseteq \Theta''$, define a partial order $\le$ on $\mathcal{T}$ as $\Theta' \le \Theta''$ iff $\Theta' \supseteq \Theta''$. Since $\emptyset \subseteq \beta^\phi(\emptyset)$ and $\Theta \supseteq \beta^\phi(\Theta)$, the function $\beta^\phi(\cdot)$ has a fixed point by Knaster-Tarski Fixed Point Theorem.[10]

Now, assume $w_{J' \cup \{i\}}(\theta) - w_{J'}(\theta) \ge w_{J \cup \{i\}}(\theta) - w_J(\theta)$ holds for all $C \subseteq J' \subseteq J \subseteq I$, $i \in I \setminus J$, and $\theta \in \Theta$. Choose ϕ with $\phi_i = 0$ for all $i \in C$. Then, we first observe that $\beta_i^\phi(\Theta') = \Theta_i$ for all $i \in C$ and $\Theta' \in \mathcal{T}$ since inequality $(*)$ implies $mp_i^J(\theta) \ge w_i(\theta_i)$ for all i, J, and θ, which in turn implies $MP_i(\theta_i; \Theta') \ge w_i(\theta_i)$ for all i, θ_i, and Θ'. Restrict the domain $\mathcal{T}$ into the collection $\mathcal{T}_C$ of subsets Θ' with $\Theta_i' = \Theta_i$ for all $i \in C$. We claim that, for all Θ', $\Theta'' \in \mathcal{T}_C$ with $\Theta' \subseteq \Theta''$, we have $\beta_i^\phi(\Theta') \supseteq \beta_i^\phi(\Theta'')$. This is obviously true for $i \in C$. For $i \in I \setminus C$, we observe that, since $C \subseteq P_{-i}(\theta; \Theta')$ for all $\Theta' \in \mathcal{T}_C$ and $P_{-i}(\theta; \Theta') \subseteq P_{-i}(\theta; \Theta'')$ for all i and θ, we have $MP_i(\theta_i; \Theta') \ge MP_i(\theta_i; \Theta'')$ by assumption. This implies that $\beta_i^\phi(\Theta') \supseteq \beta_i^\phi(\Theta'')$, that is, $\beta^\phi(\cdot)$ is monotonic on $\mathcal{T}_C$. Therefore, $\beta^\phi : \mathcal{T} \to \mathcal{T}$ has a fixed point by Knaster-Tarski Fixed Point Theorem. $\square$

For a participation fee structure $\phi = (\phi_1, \ldots, \phi_n)$ for which the hypothesis is satisfied, the theorem determines a set Θ_i^ϕ of participating types for each i. These sets are consistent in the sense that each individual's expectation of others' behaviors is rational since $\Theta^\phi \equiv (\Theta_1^\phi, \ldots, \Theta_n^\phi)$ is a fixed point of $\beta^\phi(\cdot)$. Observe that the equilibrium concept we employ here is perfect Bayesian–Nash equilibrium. In the second stage of allocation determination, it is a dominant strategy for each participant to report truthfully. On the other hand, in the first stage of participation, each individual decides to participate based on his rational expectation of others' participation decisions as well as the ensuing dominant strategy outcome in the second stage.

The budget deficit of PVCG(ϕ) (gross of participation fees) is

$$d^\phi(\theta) = - \sum_{i \in P^\phi(\theta)} \tau_i^{P^\phi(\theta)}(\theta)$$

where

$$P^\phi(\theta) \equiv \{j \in I | \theta_j \in \Theta_j^\phi\}$$

is the set of participants given the realization of $\theta \in \Theta$. So, $d^\phi(\theta) = \sum_{i \in P^\phi(\theta)}[w_{P^\phi(\theta)}(\theta) - w_{P^\phi(\theta) \setminus \{i\}}(\theta)] - w_{P^\phi(\theta)}(\theta) = \sum_{i \in P^\phi(\theta)} mp_i^{P^\phi(\theta)}(\theta) - w_{P^\phi(\theta)}(\theta)$. The net budget deficit in PVCG(ϕ) is $d^\phi(\theta) - \sum_{P^\phi(\theta)} \phi_i$.

The *ex-post efficiency loss* of PVCG(ϕ) can be defined as

$$l^\phi(\theta) \equiv w_I(\theta) - [w_{P^\phi(\theta)}(\theta) + \sum_{i \in I \setminus P^\phi(\theta)} w_i(\theta_i)].$$

Observe that we have $l^\phi(\theta) \ge 0$ for all $\theta \in \Theta$ by inequality $(*)$. Observe also that $l^\phi(\theta) = 0$ when $\phi_i = 0$ for all $i \in I$ since we have $\Theta_i^\phi = \Theta_i$ for all $i \in I$.

We introduce a parameter that captures the structure of marginal contributions and that will be used to bound the efficiency loss of the participatory VCG mechanism. For J with $C \subset J \subseteq I$, define[11]

$$\lambda \equiv \max_{C \subset J \subseteq I} \frac{E_\theta[\sum_{i \in J} mp_i^J(\theta) - w_J(\theta)]}{|J \setminus C|}.$$

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[10] Aliprantis and Border (1999) is a good reference for the statement of Knaster-Tarski Fixed Point Theorem, as well as other fixed point theorems.

[11] We will use $|A|$ to denote the number of elements in a set A.

The term $E_\theta[\sum_{i \in J} mp_i^J(\theta) - w_J(\theta)]$ is the expected difference between the sum of individual marginal contributions and the social welfare for J. The term after the max operator can be interpreted as the average expected marginal contribution of individuals in J above the social welfare. Therefore, λ is the parameter that measures the maximum of these average contributions across J. This parameter is easy to compute.

Example 2. (Continued) Consider the trading problem of Example 1. $\sum_{i \in J} mp_i^J(\theta) - w_J(\theta)$ is equal to $\max\{\theta_1, \theta_3\} - \theta_3$ when $J = \{1, 3\}$, and to $\max\{\theta_2, \theta_3\} - \theta_3$ when $J = \{2, 3\}$. Hence, $E_\theta[\sum_{i \in J} mp_i^J(\theta) - w_J(\theta)] = 1/6$ for $J = \{1, 3\}$ and $J = \{2, 3\}$. We also have $E_\theta[\sum_{i \in J} mp_i^J(\theta) - w_J(\theta)] = E_\theta[2\max\{\theta_1, \theta_2, \theta_3\} - \max\{\theta_1, \theta_3\} - \max\{\theta_2, \theta_3\}] = 1/6$ for $J = \{1, 2, 3\}$. Hence, we have $\lambda = 1/6$.

We now present the main result of this section.

Theorem 2. *Suppose* Condition 1 *holds and, moreover,* $E_\theta[\sum_{i \in C} mp_i^C(\theta) - w_C(\theta)] \leq 0$ *for the same C. Then, we can find an ex-ante BB fee structure ϕ such that the expected efficiency loss of PVCG(ϕ) is bounded by $\lambda|I \setminus C|$.*

Proof. We first note that $E_\theta[d^\phi(\theta)]$ for any ϕ is equal to

$$\sum_{J \subseteq I} p(J) E_\theta \left[\sum_{i \in J} mp_i^J(\theta) - w_J(\theta) | \theta_i \in \Theta_i^\phi \text{ for } i \in J, \theta_i \notin \Theta_i^\phi \text{ for } i \in I \setminus J \right],$$

where $p(J) \equiv \times_{i \in J} F_i(\Theta_i^\phi) \times \times_{i \in I \setminus J}[1 - F_i(\Theta_i^\phi)]$. Now consider ϕ with $\phi_i = 0$ for all $i \in C$ and $\phi_i = \lambda$ for all $i \in I \setminus C$. By Theorem 1, we can find a set Θ^ϕ of participating types. Observe that for this ϕ, we have $p(J) \neq 0$ only if $C \subseteq J$ since every individual in C always participates, that is, $\Theta_i^\phi = \Theta_i$ for all $i \in C$. Hence,

$$E_\theta[d^\phi(\theta)] = \sum_{C \subseteq J \subseteq I} p(J) E_\theta \left[\sum_{i \in J} mp_i^J(\theta) - w_J(\theta) | \theta_i \in \Theta_i^\phi \text{ for } i \in J, \theta_i \notin \Theta_i^\phi \text{ for } i \in I \setminus J \right]$$

$$\leq \sum_{C \subseteq J \subseteq I} p(J) E_\theta \left[\sum_{i \in J} mp_i^J(\theta) - w_J(\theta) \right] \leq \sum_{C \subseteq J \subseteq I} p(J) \lambda |J \setminus C|$$

$$= \sum_{C \subseteq J \subseteq I} p(J) \sum_{i \in J} \phi_i = E_\theta \left[\sum_{P^\phi(\theta)} \phi_i \right]$$

by hypothesis for C and by definition of λ for other J's. Therefore, ϕ is ex-ante BB.

Since $C \subseteq P^\phi(\theta)$ for all θ, we have

$$l^\phi(\theta) = w_I(\theta) - w_{P^\phi(\theta)}(\theta) - \sum_{i \in I \setminus P^\phi(\theta)} w_i(\theta_i) \leq \sum_{i \in I \setminus P^\phi(\theta)} [w_{P^\phi(\theta) \cup \{i\}}(\theta) - w_{P^\phi(\theta)}(\theta)]$$

$$- \sum_{i \in I \setminus P^\phi(\theta)} w_i(\theta_i) = \sum_{i \in I} \mathbf{1}[i \in I \setminus P^\phi(\theta)][w_{P^\phi(\theta) \cup \{i\}}(\theta) - w_{P^\phi(\theta)}(\theta) - w_i(\theta_i)]$$

by Condition 2 which is implied by Condition 1. Therefore, the expected efficiency loss is

$$E_\theta[l^\phi(\theta)] \leq \sum_{i \in I} E_\theta[w_{P^\phi(\theta) \cup \{i\}}(\theta) - w_{P^\phi(\theta)}(\theta) - w_i(\theta_i) | \theta_i \notin \Theta_i^\phi] = \sum_{i \in I} E_{\theta_i}[MP_i(\theta_i; \Theta^\phi)]$$

$$- w_i(\theta_i) | \theta_i \notin \Theta_i^\phi] \leq \sum_{i \in I} \phi_i = \lambda |I \setminus C|$$

by the interim individual rationality requirement for $\theta_i \notin \Theta_i^\phi$. $\square$

Remark. The hypothesis $E_\theta[\sum_{i \in C} mp_i^C(\theta) - w_C(\theta)] \leq 0$ is trivially satisfied when $C = \emptyset$ or C contains only one individual.

Remark. We can replace Condition 1 with the convexity condition and Condition 2 for the theorem.

Theorem 2 is an existence result. It states that, if the environment satisfies Condition 1, then there exists an ex-ante BB PVCG(ϕ) such that the expected efficiency loss is bounded by λ. We set the participation fees in the proof of the theorem either to λ or to zero. This ensures both that the expected deficit at the second stage is less than or equal to the expected fee collection and that the expected efficiency loss is properly bounded. We implicitly count the net budget surplus as social welfare since we do not require the expected net budget to be zero.[12] Note that, in practice, we can appropriately lower the participation fees without affecting participants' incentives in the second stage so that the expected net budget deficit is equal to zero. This will only tighten the theorem's bound for the efficiency loss for the reason that lower fees will induce more participation.

Example 3. (Continued) Consider the trading problem of Example 1. As is shown in the proof of Theorem 2, we can set $\phi_i = 1/6$ for $i = 1, 2$ and $\phi_3 = 0$. Then, every type of the seller (individual 3) participates in PVCG(ϕ), while buyers (individuals 1 and 2) participate if and only if their respective types exceed a cut-off level of $(1/3)^{1/3}$.[13] Then, PVCG(ϕ) is ex-ante BB and the expected efficiency loss is less than or equal to 1/3.

We can also use Theorem 2 to prove asymptotic efficiency. For this, consider a sequence of environments $\{\mathcal{E}^n\}_{n=1}^{\infty}$ where n is the number of individuals, and find λ^n for each $\mathcal{E}^n$. If λ^n goes to zero at a rate of $\chi(n)$, then we can find an ex-ante BB participatory VCG mechanism along $\{\mathcal{E}^n\}$ whose expected value of *per capita* efficiency loss, i.e., $E_\theta[l^\phi(\theta)]/n$, converges to zero at the rate of $\chi(n)$.

For our trading problem of Example 1, if we increase both the number of sellers and buyers simultaneously then we get that $E_\theta[l^\phi(\theta)]/n$ converges to zero at the rate of $1/n$. For a worked-out example, see Example 6 below. The next section also finds a rate of convergence for the multiple unit double auction problem in which each trader may sell or buy multiple units of a good.

3. A multiple unit double auction problem

We apply the theory developed in the previous section to a multiple unit double auction problem. We show that the participatory VCG mechanism achieves asymptotic efficiency under a weak condition. In fact, we provide a rate of convergence to full efficiency.

3.1. The environment and the mechanism

We consider a sequence of multiple unit double auction problems, $\{D^n\}_{n=1}^{\infty}$. In D^n, there is a set S^n of sellers and a set B^n of buyers. Each seller may sell up to m_s units of the good, and each buyer may buy up to m_b units of the good. We assume that m_s and m_b are independent of n.

Seller i's private information is a vector $s_i^n \equiv (s_i^n(1), \ldots, s_i^n(m_s))$, where $s_i^n(k)$ is seller i's valuation (marginal cost) for the k th unit. Likewise, buyer j's private information is a vector $b_j^n \equiv (b_j^n(1), \ldots, b_j^n(m_b))$, where $b_j^n(k)$ is buyer j's valuation (marginal benefit) for the k th unit. For $\underline{v}, \bar{v}$ in $\mathbb{R}_+$ with $\underline{v} < \bar{v}$, let

$$\Theta_s \equiv \{s = (s(1), \ldots, s(m_s)) \in \mathbb{R}_+^{m_s} | \underline{v} \leq s(1) \leq \cdots \leq s(m_s) \leq \bar{v}\},$$

and let

$$\Theta_b \equiv \{b = (b(1), \ldots, b(m_b)) \in \mathbb{R}_+^{m_b} | \bar{v} \geq b(1) \geq \cdots \geq b(m_b) \geq \underline{v}\}.$$

We assume that the set of seller i's types in D^n, denoted by $\Theta_{s,i}^n$, is a compact and convex subset of Θ_s for all n and $i \in S^n$. Likewise, we assume that the set of buyer j's types in D^n, denoted by $\Theta_{b,j}^n$, is a compact and convex subset of

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[12] Hence, the ex-ante budget balance requirement in the present paper is a weak version. It may alternatively be called as the no budget deficit requirement.

[13] The cut-off level $\hat{\theta}_i$ can be determined by equating $\hat{\theta}_i$'s expected payoff $\hat{\theta}_i \int_0^{\hat{\theta}_i} (\hat{\theta}_i - x) dx = (\hat{\theta}_i)^3/2$ from participation to the fee $\phi_i = 1/6$.

Θ_b for all n and $j \in B^n$. Hence, each seller's valuation is non-decreasing and each buyer's valuation is non-increasing. The set of type profiles in D^n is given as

$$\Theta^n \equiv \Theta^n_{s,1} \times \cdots \times \Theta^n_{s,|S^n|} \times \Theta^n_{b,1} \times \cdots \times \Theta^n_{b,|B^n|}$$

and a typical element of Θ^n is $\theta^n \equiv (\theta^n_{s,1}, \ldots, \theta^n_{s,|S^n|}, \theta^n_{b,1}, \ldots, \theta^n_{b,|B^n|})$. We denote the probability measure for seller i in D^n by F^n_i and that for buyer j by G^n_j. We assume that F^n_i's and G^n_j's are independent. (Of course, the valuations for different units of any given trader may be dependent.)

We now describe the participatory VCG mechanism for D^n for a given n (as well as other elements of the environment). The underlying VCG mechanism is the one discussed in Vickrey (1961). In the following description, we will dispense with the superscript n to avoid notational congestion. First of all, we emphasize that the autarky payoff level $w_i(\theta_i)$ is given as $\sum_{k=1}^{m_s} s_i(k)$ for seller $i \in S$, and $A_j(\theta_j) = 0$ for all buyer $j \in B$. That is, seller i can achieve a payoff level of $\sum_{k=1}^{m_s} s_i(k)$ in isolation, while buyer j can only achieve zero payoff.[14] For a given participation fee structure $\phi = (\phi_{s,1}, \ldots, \phi_{s,|S|}, \phi_{b,1}, \ldots, \phi_{b,|B|})$, let $S_p \subseteq S$ and $B_p \subseteq B$ denote the realized set of participating sellers and buyers, respectively. We need the participation fees to ensure ex-ante budget balance: the mechanism always runs a budget deficit due to its payment rule as described below. With positive fees, some traders may not find it profitable to participate since the expected net gain from trade may be less than the participation fee. Since PVCG(ϕ) is DIC for the participants, each seller $i \in S_p$ submits his true valuation vector $s_i = (s_i(1), \ldots, s_i(m_s))$ as his offer vector and each buyer $j \in B_p$ submits her true valuation vector $b_j = (b_j(1), \ldots, b_j(m_b))$ as her bid vector.

Suppose the set of participating traders and the offers/bids of the participants are given.[15] Sellers' offers can be arrayed in increasing order as $s_{(1)} \leq \cdots \leq s_{(m_s|S_p|)}$, and buyers' bids can be arrayed in decreasing order as $b_{(1)} \geq \cdots \geq b_{(m_b|B_p|)}$. Note that we have $m_s|S_p|$ offers since there are $|S_p|$ participating sellers and each seller submits m_s offers. Likewise, we have $m_b|B_p|$ bids since there are $|B_p|$ participating buyers and each buyer submits m_b bids. Alternatively, we can array all the offers and bids in decreasing order as $v_{(1)} \geq \cdots \geq v_{(m_s|S_p|+m_b|B_p|)}$.[16] We define several sets. The set V is a set of offers and bids, that is,

$$V \equiv \{v_{(1)}, \ldots, v_{(m_s|S_p|+m_b|B_p|)}\}.$$

(We regard each offer or bid as a distinct object. Therefore, there are exactly $m_s|S_p| + m_b|B_p|$ elements in V even when there are ties. The same convention applies for sets defined below.) Let

$$V_H \equiv \{v_{(1)}, \ldots, v_{(m_s|S_p|)}\} \quad \text{and} \quad V_L \equiv \{v_{(m_s|S_p|+1)}, \ldots, v_{(m_s|S_p|+m_b|B_p|)}\}$$

denote the set of higher and lower valuations, respectively. Note that $m_s|S_p|$ is the total units of the good for trade. In addition, let

$$R_{s,i} \equiv \{s_i(1), \ldots, s_i(m_s)\} \quad \text{and} \quad R_{b,j} \equiv \{b_j(1), \ldots, b_j(m_b)\}$$

denote the set of valuations for each $i \in S_p$ and $j \in B_p$, respectively, and let

$$R_s \equiv \cup_{i \in S_p} R_{s,i} \quad \text{and} \quad R_b \equiv \cup_{j \in B_p} R_{b,j}$$

denote the set of participating sellers' and buyers' valuations, respectively. Finally, let

$$R \equiv R_s \cup R_b.$$

Then, we can define the obvious matching bijection

$$\mu : R \rightarrow V.$$

The set Y_J of all possible outcomes achievable among J when the set of participants is $J = S_p \cup B_p$ is the collection of all feasible allocations of $m_s|S_p|$ units of the good to the participants in J. The units will be allocated to those who

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[14] We can alternatively set, as other papers do, the autarky payoff levels for all traders equal to zero. In this case, we should subtract sellers' valuations for the *traded* items from the social welfare below. Note that this is only a matter of normalization. We adopt the normalization in the text since it is convenient. It is also natural given that we view $s_i(k)$ as the utility seller i obtains from the k th unit of the good.

[15] Note that the description obviously includes the case when every trader participates, i.e., the case when $S_p = S$ and $B_p = B$.

[16] In the order statistics above, ties can be broken in any predetermined way.

value them most in an efficient allocation. Therefore, the social welfare from an efficient allocation for J when the type profile is θ is $w_J(\theta) = \sum_{k=1}^{m_s|S_p|} v_{(k)}$. The volume of trade for $i \in S_p$ in PVCG(ϕ) is given as

$$q_{s,i} \equiv |\mu(R_{s,i}) \cap V_L|,$$

i.e., the number of his offers less than or equal to $v_{(m_s|S_p|+1)}$. His receipt of money for the sale of goods is

$$p_{s,i} \equiv \min\{v_1 + \cdots + v_{q_{s,i}}| \text{ each } v_k \text{ is distinct and } v_k \in \mu(R \setminus R_{s,i}) \cap V_H\}$$

In words, his receipt of money for the first unit he sells is the lowest among the offers/bids other than his own offers which are greater than or equal to $v_{(m_s|S_p|)}$, his receipt of money for the second unit he sells is the second lowest among the offers/bids other than his own offers which are greater than or equal to $v_{(m_s|S_p|)}$, and so on up to $q_{s,i}$. In a symmetric way, the volume of trade for $j \in B_p$ is given as

$$q_{b,j} \equiv |\mu(R_{b,j}) \cap V_H|,$$

and her payment of money is

$$p_{b,j} \equiv \max\{v_1 + \cdots + v_{q_{b,j}}| \text{ each } v_k \text{ is distinct and } v_k \in \mu(R \setminus R_{b,j}) \cap V_L\}.$$

The aggregate volume of trade is $q = \sum_{i \in S_p} q_{s,i} = \sum_{j \in B_p} q_{b,j}$. (Note that $\sum_{i \in S_p} q_{s,i} = |\mu(R_s) \cap V_L| = |V_L| - |\mu(R_b) \cap V_L| = |V_L| - |\mu(R_b)| + |\mu(R_b) \cap V_H| = m_b|B_p| - m_b|B_p| + |\mu(R_b) \cap V_H| = \sum_{j \in B_p} q_{b,j}$.)   The budget deficit is $\sum_{i \in S_p} p_{s,i} - \sum_{j \in B_p} p_{b,j} \geq 0$.

In the special case of unit supply and unit demand (that is, when $m_s = m_b = 1$), a seller trades if and only if his offer is less than or equal to $v_{(m_s|S_p|+1)}$, and his receipt is $v_{(m_s|S_p|)}$. Likewise, a buyer trades if and only if her bid is greater than or equal to $v_{(m_s|S_p|)}$, and her payment is $v_{(m_s|S_p|+1)}$. We can easily show that $v_{(m_s|S_p|)}$ is equal to $\min\{b_{(q)}, s_{(q+1)}\}$ and $v_{(m_s|S_p|+1)}$ is equal to $\max\{s_{(q)}, b_{(q+1)}\}$ in this case.

We show in the following lemma that the mechanism just described is indeed a participatory VCG mechanism of the previous section.

Lemma 1.   *The mechanism above for the double auction problem is a participatory VCG mechanism.*

Proof.   It is obvious that the mechanism achieves outcome efficiency for any given set J of participants. Next, we check the mechanism's transfer rule. Seller i's payoff from the outcome, $u_i(\alpha_J(\theta); \theta_i)$, is $\sum_{v_k \in \mu(R_{s,i}) \cap V_H} v_k$, and buyer j's payoff from the outcome, $u_j(\alpha_J(\theta); \theta_j)$, is $\sum_{v_k \in \mu(R_{b,j}) \cap V_H} v_k$. Therefore,

$$w_{S_p \cup B_p}(\theta) - u_i(\alpha_J(\theta); \theta_i) = \sum_{v_k \in V_H} v_k - \sum_{v_k \in \mu(R_{s,i}) \cap V_H} v_k$$

for seller $i \in S_p$ and

$$w_{S_p \cup B_p}(\theta) - u_j(\alpha_J(\theta); \theta_j) = \sum_{v_k \in V_H} v_k - \sum_{v_k \in \mu(R_{b,j}) \cap V_H} v_k$$

for buyer $j \in B_p$. On the other hand, observe that

$$w_{S_p \cup B_p \setminus \{i\}}(\theta) = \sum_{v_k \in V_H} v_k - \sum_{v_k \in \mu(R_{s,i}) \cap V_H} v_k - p_{s,i}$$

for seller $i \in S_p$ and

$$w_{S_p \cup B_p \setminus \{j\}}(\theta) = \sum_{v_k \in V_H} v_k - \sum_{v_k \in \mu(R_{b,j}) \cap V_H} v_k + p_{b,j}$$

for buyer $j \in B_p$. Therefore, $\tau_i^{S_p \cup B_p}(\theta) = -p_{s,i}$ for seller $i \in S_p$ and $\tau_j^{S_p \cup B_p}(\theta) = p_{b,j}$ for buyer $j \in B_p$. This confirms that the mechanism is indeed a participatory VCG mechanism. $\square$

We add another lemma, which is needed in Theorem 3.

Lemma 2. *For any given set $J = S_p \cup B_p$ of participants, we have*

$$\sum_{i \in S_p} mp_i^J(\theta) + \sum_{j \in B_p} mp_j^J(\theta) - w_J(\theta) \leq q[v_{(m_s|S_p|-m_s+1)} - v_{(m_s|S_p|+m_b)}].$$

Proof. For $i \in S_p$, we have $mp_i^J(\theta) = w_J(\theta) - w_{J\setminus\{i\}}(\theta) = \sum_{v_k \in V_H} v_k - \sum_{v_k \in V_H} v_k + \sum_{v_k \in \mu(R_{s,i}) \cap V_H} v_k + p_{s,i} \leq \sum_{v_k \in \mu(R_{s,i}) \cap V_H} v_k + \sum_{k=1}^{q_{s,i}} v_{(m_s|S_p|-m_s+k)}$. For $j \in B_p$, we have $mp_j^J(\theta) = w_J(\theta) - w_{J\setminus\{j\}}(\theta) = \sum_{v_k \in V_H} v_k - \sum_{v_k \in V_H} v_k + \sum_{v_k \in \mu(R_{b,j}) \cap V_H} v_k - p_{b,j} \leq \sum_{v_k \in \mu(R_{b,j}) \cap V_H} v_k - \sum_{k=1}^{q_{b,j}} v_{(m_s|S_p|+m_b+1-k)}$. Therefore, $\sum_{i \in S_p} mp_i^J(\theta) + \sum_{j \in B_p} mp_j^J(\theta) - w_J(\theta) \leq \sum_{v_k \in V_H} v_k + \sum_{i \in S_p} \sum_{k=1}^{q_{s,i}} v_{(m_s|S_p|-m_s+k)} - \sum_{j \in B_p} \sum_{k=1}^{q_{b,j}} v_{(m_s|S_p|+m_b+1-k)} - \sum_{v_k \in V_H} v_k \leq q[v_{(m_s|S_p|-m_s+1)} - v_{(m_s|S_p|+m_b)}]$. $\square$

3.2. Asymptotic efficiency of the participatory VCG mechanism

In the remainder of the paper, we will show that asymptotic efficiency (and a rate of convergence) of the multiple unit double auction problem can be determined by looking at the probability measures F_i's and G_j's on the type space. Let $s_{(1)}^n \leq \cdots \leq s_{(m_s|S^n|)}^n$ and $b_{(1)}^n \geq \cdots \geq b_{(m_b|B^n|)}^n$ be the order statistics of sellers' and buyers' valuations in D^n. We will assume that the distance between these valuations gets smaller at a specified rate as n increases.

Condition 3. *There is a nonnegative function $\chi(\cdot)$ with $\lim_{n\to\infty} \chi(n) = 0$ such that* [17]

$$\min\{\max_k E[s_{(k+m_s+m_b-1)}^n - s_{(k)}^n], \quad \max_k E[b_{(k)}^n - b_{(k+m_s+m_b-1)}^n]\} \leq \chi(n).$$

As stated, either sellers' valuations or buyers' valuations get closer at the rate of $\chi(n)$ as n increases. In the double auction context, this implies that each trader's influence on the terms of trade as well as his marginal contribution to the social welfare gets smaller. Note that, when the valuations are spread all over the interval $[\underline{v}, \bar{v}]$, this condition is equivalent to the condition that the number of valuations in any given interval $[\underline{w}, \bar{w}] \subseteq [\underline{v}, \bar{v}]$ increases on the order of $1/\chi(n)$. Condition 3 is quite weak: it allows atomic probability measures, asymmetry across individuals, etc. We provide some examples below.

Example 4. (i.i.d. distributions) Let both $|S^n|$ and $|B^n|$ increase at the rate of n. Assume that $F_i^n = F$ and $G_j^n = G$ for all i, j and n, and that the density functions f and g exist and are bounded away from zero. Then, we have $\chi(n) = 1/n$.

Observe that the order statistics $\{s_{(k)}^n, \ldots, s_{(k+m_s+m_b-1)}^n\}$ include at least two sellers' valuations since $m_s + m_b > m_s$. Likewise, the order statistics $\{b_{(k)}^n, \ldots, b_{(k+m_s+m_b-1)}^n\}$ include at least two buyers' valuations. Then, since the valuation vectors are independent across traders, it is a straightforward exercise to derive that both $E[s_{(k+m_s+m_b-1)}^n - s_{(k)}^n]$ and $E[b_{(k)}^n - b_{(k+m_s+m_b-1)}^n]$ are of order $1/n$. [18]

Example 5. (Atomic distributions) Assume that each individual's distribution function, F_i^n or G_j^n, puts all the weight on a single valuation vector. Then, Condition 3 is easily satisfied as long as these atoms get close as n increases.

The previous example admits asymmetry of distribution across traders. It is also a straightforward matter to construct general asymmetric distributions that satisfy Condition 3. We now present the main result of this section.

Theorem 3. *Consider the multiple unit double auction problems, $\{D^n\}_{n=1}^{\infty}$. If Condition 3 holds, then there exists an ex-ante BB participatory VCG mechanism along $\{D^n\}$ such that the expected per capita efficiency loss converges to zero at the rate of $\chi(n)$.*

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[17] Note that the max is taken for $1 \leq k \leq m_s|S^n| - m_s - m_b + 1$ in the first case, and for $1 \leq k \leq m_b|B^n| - m_s - m_b + 1$ in the second case. If $m_s|S^n| < m_s + m_b$ then we set the first max to be ∞, and likewise if $m_b|B^n| < m_s + m_b$ then we set the second max to be ∞. We restrict our attention to cases when either $m_s|S^n| \geq m_s + m_b$ or $m_b|B^n| \geq m_s + m_b$. Observe that this is satisfied if $|S^n| \geq 2$ and $|B^n| \geq 2$.

[18] See, for example, David (1981). See also Yoon (2001).

Proof. Fix D^n and let $C = S^n$ if $\max_k E[s^n_{(k+m_s+m_b-1)} - s^n_{(k)}] \leq \max_k E[b^n_{(k)} - b^n_{(k+m_s+m_b-1)}]$; otherwise let $C = B^n$. In what follows, we will assume that $C = S^n$ without loss of generality: the other case follows essentially symmetric arguments. We first show that D^n satisfies Condition 1, which guarantees the existence of a consistent set Θ^ϕ of participating types. Choose subsets J, J' with $C \subseteq J' \subseteq J \subseteq I$. Note that all the sellers are in J' and J. Let $v^{J'}_{(1)} \geq \cdots \geq v^{J'}_{(m_s|C|+m_b|J'\setminus C|)}$ and $v^J_{(1)} \geq \cdots \geq v^J_{(m_s|C|+m_b|J\setminus C|)}$ be the order statistics derived from offers/bids of J' and J, respectively. Consider $i \in I \setminus J$, and let $v^i_{(1)} \geq \cdots \geq v^i_{(m_b)}$ be the order statistics of i's bids. Define $\bar{k}^{J'} = \max_k\{v^i_{(k)} \geq v^{J'}_{(m_s|C|+1-k)}\}$ and $\bar{k}^J = \max_k\{v^i_{(k)} \geq v^J_{(m_s|C|+1-k)}\}$. Since $J' \subseteq J$, we have $v^J_{(k)} \geq v^{J'}_{(k)}$ for all $k = 1, \ldots, m_s|C| + m_b|J' \setminus C|$ and hence $\bar{k}^J \leq \bar{k}^{J'}$. Therefore, $w_{J' \cup \{i\}}(\theta) - w_{J'}(\theta) = \sum_{k=1}^{\bar{k}^{J'}}[v^i_{(k)} - v^{J'}_{(m_s|C|+1-k)}] \geq \sum_{k=1}^{\bar{k}^J}[v^i_{(k)} - v^J_{(m_s|C|+1-k)}] = w_{J \cup \{i\}}(\theta) - w_J(\theta)$.

Next, by Lemma 2 and the fact that $q \leq m_b|J \setminus C|$ for all J with $C \subseteq J \subseteq I$, we have

$$\lambda^n \leq m_b \max_{C \subset J \subseteq I} E[v_{(m_s|S^n|-m_s+1)} - v_{(m_s|S^n|+m_b)}],$$

where recall that λ^n is defined to be $\max_{C \subset J \subseteq I} E_\theta[\sum_{i \in J} mp^J_i(\theta) - w_J(\theta)]/|J \setminus C|$.

We have shown that each D^n satisfies Condition 1. In addition, since $C = S^n$ so that C includes no buyer, we have $\sum_{i \in C} mp^C_i(\theta) = w_C(\theta)$ for all θ. Theorem 2 then implies that, for each D^n, there exists a participatory VCG mechanism which is ex-ante BB and whose expected per capita efficiency loss is bounded by λ^n. Since λ^n converges to 0 at the rate of $\chi(n)$ by Condition 3, we are done. $\square$

We provide an example to assist readers' understanding of the theorem.

Example 6. There are n sellers and n buyers in D^n. Each seller owns one unit of a good, while each buyer may buy up to one unit of the good. Each individual's type θ_i is independently drawn from the uniform distribution on the unit interval. We set C to be the set of sellers. Thus, the participation fees are such that $\phi_i = 0$ for every seller $i \in S^n$ and $\phi_j = 1/(n+1)$ for every buyer $j \in B^n$. Then, we have $E_\theta[\sum_{i \in J} mp^J_i(\theta) - w_J(\theta)] \leq q/(n+1)$. Hence, the expected per capita efficiency loss converges to zero at the rate of $1/n$.

Note that this example is a simple example for the unit supply-unit demand case. What we have proven in this section is in fact an asymptotic efficiency result for the multiple unit double auction problem, where each trader may trade multiple units. As noted in the introduction, no paper has previously investigated the multiple unit double auction problem.[19] On the contrary, several mechanisms have been proposed and analyzed for the unitary case: there are k-double auctions (Gresik and Satterthwaite, 1989; Rustichini et al., 1994 among others), McAfee's (1992) dominant strategy double auction, the fixed-price mechanism (Hagerty and Rogerson, 1987), and the modified Vickrey double auction (Yoon, 2001). Among these, k-double auctions have the fastest convergence rate of $1/n^2$ for the uniform distribution.[20]

4. Discussion

We have introduced the participatory VCG mechanism and analyzed its properties. The participatory VCG mechanism is conceived to overcome the well-known weakness of the original VCG mechanism. Despite of its prominence in the mechanism design theory, the VCG mechanism has a severe drawback for many interesting problems such as double auctions and public good provision: it does not in general satisfy both ex-ante budget balance and interim individual rationality requirements.

The participatory VCG mechanism is a modified version of the VCG mechanism such that ex-ante budget balance and interim individual rationality are satisfied. The way it achieves this is to place, before the allocation stage, the voluntary participation stage in which the mechanism collects the fees that will be used to cover the deficit and the individuals decide whether to participate in the mechanism.

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[19] In a recent paper, after the working paper version of the present paper was circulated, Cripps and Swinkels (2003) generalized k-double auctions to very general multiple unit environments and proved that inefficiency disappears at rate $1/n^{2-\alpha}$ for any $\alpha > 0$.

[20] Satterthwaite and Williams (2002) established that k-double auctions are *worst-case asymptotic optimal* for the unit supply-unit demand case.

The main result of this paper is the existence of a participatory VCG mechanism that satisfies ex-ante budget balance and interim individual rationality, whose expected efficiency loss is bounded by a parameter that captures the structure of individuals' marginal contributions to the social welfare. This cures the major flaw of the VCG mechanism and makes it a feasible mechanism for most economic environments. We then applied the theory to multiple unit double auction problems to illustrate that the participatory VCG mechanism achieves asymptotic efficiency for quite general environments.

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