Journal of Economic Theory **101**, 572–584 (2001)
doi:10.1006/jeth.2000.2736, available online at http://www.idealibrary.com on IDEAL®

NOTES, COMMENTS, AND LETTERS TO THE EDITOR

The Modified Vickrey Double Auction[1]

Kiho Yoon

Department of Economics, Sogang University, 1 Shinsu-dong, Mapo-gu, Seoul, Korea 121-742
kiho@ccs.sogang.ac.kr

Received April 19, 2000; final version received August 21, 2000; published online April 26, 2001

We modify W. Vickrey's (1961, *J. Finance* **16**, 8–37) mechanism for call markets by introducing the participation stage and study the efficiency properties of the modified mechanism. We provide sufficient conditions under which the modified Vickrey double auction achieves full efficiency. In addition, we prove that the modified Vickrey double auction achieves asymptotic efficiency even when full efficiency cannot be achieved. *Journal of Economic Literature* Classification Numbers: C72, D44, D82. © 2001 Elsevier Science

Key Words: double auction; multilateral bargaining; efficiency; dominant strategy.

1. INTRODUCTION

Vickrey's celebrated paper [13] starts with a graphical discussion of a mechanism for call markets where sellers submit their privately known supply curves and buyers submit their privately known demand curves to an "exclusive public marketing agency." Vickrey, however, does not pursue this line of research further, but instead spares most of his pages for the discussion of auction mechanisms where there is only one seller who owns one indivisible item to be sold. Consequently, compared to the development of auction literature in the following decades, his discussion about double auction mechanism has not attracted much attention. In this paper, we (i) modify the Vickrey double auction mechanism by introducing the participation stage, and (ii) study the efficiency properties of the modified

[1] I thank Leonid Hurwicz, James Jordan, Andrew McLennan, Steven Williams, seminar participants at the University of Minnesota, Hong Kong University of Science and Technology, and Universidad Carlos III de Madrid. I also thank an anonymous referee and an Associate Editor of this journal for helpful suggestions.

0022-0531/01 $35.00

mechanism. We study the mechanism under the environment where each seller has one indivisible item to sell and each buyer wants to buy at most one item. We find that the Vickrey double auction is a direct generalization of Vickrey's second price auction to markets with both sides in the sense that there are two prices, the seller price and the buyer price, each of which can be interpreted as a second price.

Given the fixed set of potential sellers and buyers, Vickrey's proposed double auction mechanism is designed to achieve two properties, namely, efficiency and dominant strategy incentive compatibility. This mechanism, however, does not satisfy both budget balance and individual rationality if taken in its original form as he discusses. That is, the public marketing agency runs a deficit to achieve efficiency for the whole set of traders. One remedy to this problem is to collect participation fees from those traders who want to enter the mechanism. The main contribution of this paper is to explicitly analyze the entry decision of traders by introducing the participation stage, and study how well the modified Vickrey double auction performs in achieving efficiency under the additional requirement of budget balance and individual rationality.

We first provide sufficient conditions on the distributions of traders' valuations under which the modified Vickrey double auction satisfies efficiency, dominant strategy incentive compatibility, ex-ante budget balance, and interim individual rationality. The next result is about asymptotic efficiency. We show that, even though the traders' valuations are such that full efficiency cannot be achieved, the expected efficiency loss is bounded. This bound depends only on a parameter of the distribution functions, not on the number of traders. Consequently, the modified Vickrey double auction achieves asymptotic efficiency since the efficiency loss per trader is asymptotically zero.[2]

The research on the efficiency of markets with private information has a long history and is too vast for us to overview the whole subject here. Therefore, we will discuss only a subset of the literature which is directly related to this paper, namely the double auction literature or the multilateral bargaining literature. Since the pioneering works of Chatterjee and Samuelson [1] and Myerson and Satterthwaite [9] on the bilateral bargaining situation where one seller has one indivisible item to sell and one buyer wants to buy that item, a series of papers have studied the multilateral bargaining situation, and various rules of trading have been

[2] These positive results are particularly interesting in view of Vickrey's pessimism. He notes, "It is tempting to try to modify this scheme in various ways that would reduce or eliminate this cost of operation while still preserving the tendency to optimum allocation of resources. However, it seems that all modifications that do diminish the cost of the scheme either imply the use of some external information as to the true equilibrium price or reintroduce a direct incentive for misrepresentation of the marginal-cost or marginal-value curves."

analyzed. There are k-double auctions (Wilson [15], Gresik and Satterthwaite [3], Leninger *et al.* [6], Satterthwaite and Williams [11, 12], and Rustichini *et al.* [10]), the fixed price mechanism (Hagerty and Rogerson [5]), and McAfee's dominant strategy double auction [8]. Of all these mechanisms, the modified Vickrey double auction is the only one that can achieve full efficiency. In addition, honesty is the only dominant strategy in the modified Vickrey double auction.[3] Therefore, the complexity of traders' decisions is dramatically reduced.[4]

2. MAIN RESULTS

2.1. *The Mechanism*

There is a set $\mathcal{S} = \{1, ..., M\}$ of sellers and a set $\mathcal{B} = \{1, ..., N\}$ of buyers for a good. Each seller has one indivisible item to sell and each buyer wants to buy at most one item. Seller i's privately known valuation for the good is denoted by c_i, and buyer j's privately known valuation for the good is denoted by b_j. We impose the independence assumption throughout this paper.

Assumption (Independence). (i) Each seller's (buyer's) valuation is drawn independently according to the distribution $F(G)$ on the interval $[\underline{s}, \bar{s}]$ $([\underline{b}, \bar{b}])$, with the density $f(g)$ bounded away from 0.

(ii) $[\underline{s}, \bar{s}]$ and $[\underline{b}, \bar{b}]$ intersect with a non-empty interior.

Let $\gamma_s = \inf\{f(s) : s \in [\underline{s}, \bar{s}]\}$ and $\gamma_b = \inf\{g(b) : b \in [\underline{b}, \bar{b}]\}$. By the Assumption, we have $\gamma_s > 0$ and $\gamma_b > 0$.

There is a third party who controls all the trades, whom we call the *market maker*.[5] Sellers and buyers first decide whether to participate in the mechanism by agreeing to pay a fee to the market maker. The participation fee for seller i is denoted by ϕ_i^s and the participation fee for buyer j is denoted by ϕ_j^b. Let the set of participating sellers (buyers) be $S \subseteq \mathcal{S}$ $(B \subseteq \mathcal{B})$, and let m (n) be the number of participating sellers (buyers), i.e., $|S| = m$ $(|B| = n)$. Each participating seller, say seller $i \in S$, submits an offer s_i, and each participating buyer, say buyer $j \in B$, submits a bid b_j.

The volume and terms of trade of the modified Vickrey double auction is determined as follows. Sellers' offers are arrayed in increasing order, resulting in the order statistics $s_{(1)} \leqslant s_{(2)} \leqslant \cdots \leqslant s_{(m)}$. Similarly, buyers' bids

[3] McAfee's mechanism has this property, too.
[4] The entry decision is not trivial, however.
[5] Vickrey calls this entity as the exclusive public marketing agency.

are arrayed in decreasing order, resulting in the order statistics $b_{(1)} \geqslant b_{(2)} \geqslant \cdots \geqslant b_{(n)}$. If there is a tie, then it can be broken in any predetermined way. For convenience, we will follow the convention that $s_{(0)} = b_{(n+1)} = \min\{\underline{s}, \underline{b}\}$, and $b_{(0)} = s_{(m+1)} = \max\{\bar{s}, \bar{b}\}$. The volume of trade is determined as the number k such that $s_{(k)} \leqslant b_{(k)}$ and $s_{(k+1)} > b_{(k+1)}$. That is, the volume of trade k is $\max\{\kappa : s_{(\kappa)} \leqslant b_{(\kappa)}\}$. A seller whose offer is one of $s_{(1)}, ..., s_{(k)}$ and a buyer whose bid is one of $b_{(1)}, ..., b_{(k)}$ can trade, and other participants cannot.

Each successful seller receives the same amount of money from the market maker, and the seller price is set as $p^s = \min\{b_{(k)}, s_{(k+1)}\}$. Similarly, each successful buyer pays the same amount of money to the market maker, and the buyer price is set as $p^b = \max\{s_{(k)}, b_{(k+1)}\}$. The seller price p^s is set to the best unsuccessful offer $s_{(k+1)}$ as long as this is lower than the worst successful bid $b_{(k)}$; otherwise it is set to the worst successful bid. Likewise, the buyer price p^b is set to the best unsuccessful bid $b_{(k+1)}$ as long as this is higher than the worst successful offer $s_{(k)}$; otherwise it is set to the worst successful offer. In this respect, this mechanism is a generalization of Vickrey's second-price auction to markets with both sides.[6] It is easy to see that $p^b \leqslant p^s$. Note that seller i's utility is $p^s - c_i - \phi_i^s$ if he participates in the mechanism and sells his item; $c_i - \phi_i^s$ if he participates in the mechanism and is unsuccessful; and c_i if he does not participate. Likewise, buyer j's utility is $v_j - p^b - \phi_j^b$ if she participates in the mechanism and obtains one item; $-\phi_j^b$ if she participates in the mechanism and is unsuccessful; and 0 if she does not participate.

2.2. *Sufficient Conditions for Efficiency*

Vickrey's fundamental insight was that it is each *particpant's* dominant strategy in this mechanism to report his/her own true valuation. That is, honesty is each participant's dominant stragey for any sets S and B of participating sellers and buyers. Therefore, this mechanism is dominant strategy incentive compatible, and also efficient in the sense that all the potential gains from trade are realized. In fact, it is straightforward to check that the original Vickrey double auction without the participation stage is a member of the Vickrey–Clarke–Groves mechanisms.[7]

Dominant strategy incentive compatible and efficient mechanisms do not generally satisfy other desirable properties, like budget balance and

[6] The receipts of the successful sellers and the payments of the successful buyers are individualized for the general cases when sellers and buyers submit their respective supply and demand schedules for multiple units. A formal treatment of the general case can be found in an earlier version of this paper.

[7] See Makowski and Ostroy [7] for a precise definition. Refer also to Clarke [2] and Groves [4], in addition to Vickrey [13].

individual rationality. In our mechanism, the sum of participating sellers' receipts exceeds the sum of participating buyers' payments since $p^b \leqslant p^s$. So budget balance will not be satisfied unless the market maker charges strictly positive fees to some participants. Then some traders may not enter the mechanism since they may find that their expected gains from trade do not cover the fee. Therefore, if we impose the budget balance requirement, then, although this mechanism is efficient for the participants, it may not be efficient for the whole set $\mathscr{S} \cup \mathscr{B}$ of traders. The main contribution of this paper is to study the properties of Vickrey double auction with the entry decision explicitly modeled. In other words, we analyze the efficiency properties of the Vickrey double auction with the additional requirement of budget balance and individual rationality. We first provide sufficient conditions for efficiency.

THEOREM 1. *The modified Vickrey double auction satisfies efficiency, dominant strategy incentive compatibility, ex-ante budget balance, and interim individual rationality if one of the following conditions holds.*[8]

(i) *$M > N$ and*

$$\sum_{r=N}^{M} \binom{M}{r} \int_{\underline{s}}^{\underline{b}} [F(s)]^r [1-F(s)]^{M-r} \, ds \geqslant \frac{1}{\gamma_s (M+1)}.$$

(ii) *$N > M$ and*

$$\sum_{r=0}^{N-M} \binom{N}{r} \int_{\bar{s}}^{\bar{b}} [G(b)]^r [1-G(b)]^{N-r} \, db \geqslant \frac{1}{\gamma_b (N+1)}.$$

Proof. See the Appendix. ▮

When one of the conditions in Theorem 1 holds, then the market maker can charge ex-ante budget balancing fees in a way that all the potential traders enter the mechanism. Therefore, efficiency is achieved for $\mathscr{S} \cup \mathscr{B}$. Theorem 1 is a straightforward extension of one of Williams's results [14, Theorem 4], which discusses conditions for the existence of an efficient, Bayesian incentive compatible, ex-ante budget balancing, and interim individually rational mechanism for the double auction environment. Using the formulas derived in Williams [14], Theorem 1 provides testable conditions in terms of the distributions. Note that the conditions in the theorem are easily satisfied for many distributions.

[8] The modified Vickrey double auction satisfies ex-ante budget balance if, for any $S \subseteq \mathscr{S}$ and $B \subseteq \mathscr{B}$, $E[k(p^s - p^b)] \leqslant \sum_{i \in S} \phi_i^s + \sum_{j \in B} \phi_j^b$.

EXAMPLE. F is a uniform distribution over $[\underline{s}, \bar{s}] = [0, 1]$, and $\underline{b} = 0.5$. When $M = 5$ and $N = 2$, the first condition is satisfied since *LHS* is 0.1875, and *RHS* is 0.1667.

2.3. *The Rate of Convergence to Efficiency*

Even when full efficiency cannot be achieved,[9] the expected efficiency loss of the modified Vickrey double auction is bounded. Let $I(M, N)$ be the expected efficiency loss (that is, the expected value of the unrealized gains from trade)[10] of the mechanism under the requirement of ex-ante budget balance and interim individual rationality. We have

THEOREM 2. *For* $M, N \geqslant 2$,

$$I(M, N) \leqslant \frac{1}{\gamma_s} 1_{\{M \geqslant N\}} + \frac{1}{\gamma_b} 1_{\{M \leqslant N\}}.$$

Proof. See the Appendix. ∎

Therefore, the expected efficiency loss per trader is of order $1/(M+N)$. Although this theoretical bound for the rate of convergence to efficiency looks a little bit loose, the simulation result presented below suggests that the modified Vickrey double auction does not have a rate of order $1/(M+N)^2$, as the k-double auction and McAffee's dominant strategy double auction have.

We provide a simulation result for the case when there are equal numbers of sellers and buyers, and sellers' and buyers' valuations are drawn from the uniform distribution on the unit interval. We compare the rate of convergence of the modified Vickrey double auction with those of other mechanisms in the literature. As seen from Table I, the modified Vickrey double auction performs worse than the k-double auction for this particular case. Though we provide the simulation result for this case in order to compare with existing simulation results, we want to emphasize that this does not necessarily imply that the modified Vickrey double auction performs worse generally. To make the point obvious, consider the case when the sufficient condition for full efficiency is satisfied. Then the modified Vickrey double auction performs definitely better then the k-double auction and McAfee's dominant strategy double auction, since the latter mechanisms cannot achieve full efficiency by their rules.

[9] For example, if $[\underline{s}, \bar{s}] = [\underline{b}, \bar{b}]$ then we cannot achieve full efficiency for any finite M and N.
[10] See formula (7) below for a precise definition.

TABLE I

Relative Inefficiency of Different Mechanisms: Uniform Case

$M = N$	MVDA	0.5-DAL	0.5-DAM	DSA
2	11.938	5.6	6.3	17.741
3	6.905	n.a.[a]	n.a.	11.079
4	4.572	1.5	1.6	7.593
5	3.298	n.a.	n.a.	5.417
8	1.578	0.39	0.39	n.a.
10	1.119	n.a.	n.a.	1.625
25	0.268	n.a.	n.a.	0.295
50	0.087	n.a.	n.a.	0.076
100	0.029	n.a.	n.a.	0.019

[a] n.a., not available.

Note. MVDA is the relative inefficiency of the modified Vickrey double auction when the participation fee is charged to both sides. The 0.5-DAL and 0.5-DAM are the relative inefficiencies of the least and the most inefficient equilibria of the 0.5-double auction, and DSA is the relative inefficiency of McAfee's dominant strategy double auction. MVDA is obtained by simulating 500,000 times for each $M = N$. The 0.5-DAL, 0.5-DAM, and DSA are from the corresponding papers.

In Table I, the relative inefficiency is the percentage value of the ratio of the total gains from trade unfulfilled under the mechanism to the total possible gains from trade. The column MVDA reports the relative inefficiency of the modified Vickrey double auction when the participation fee is charged to both sides. The columns 0.5-DAL and 0.5-DAM are the relative inefficiencies of the least and the most inefficient equilibria of the 0.5-double auction analyzed in Rustichini *et al.* [10], and the column DSA is the relative inefficiency of the dominant strategy double auction of McAfee [8]. As Table I shows, the modified Vickrey double auction performs better than McAfee's mechanism for small numbers of traders, but does not do as well as the 0.5-double auction. Still, the inefficiency of the modified Vickrey double auction is quite negligible for large $M = N$.

3. A FINAL COMMENT

We defined ex-ante budget balance in the sense that the market maker's net payment after the fee collection is non-positive in expectation. (Please refer to footnote 8.) Alternatively, we can require that the market maker's net payment must be equal to zero in expectation. There are two implications from this change. First, the fee must be charged correctly reflecting the distribution functions, F, G under the strong version, i.e., the case when the equality holds, while the market maker has some freedom over the actual level of the fee under the weak version. Second, we implicitly count the market maker's profit (the negative of the net payment) as welfare under the weak version.

APPENDIX

Proof of Theorem 1. We first derive a bound on the expected difference of the seller price and the buyer price.[11]

LEMMA 1. (i) *If* $M > N$, *then* $E[p^s_{(M,N)} - p^b_{(M,N)}] \leqslant (\gamma_s(M+1))^{-1}$.

(ii) *If* $M < N$, *then* $E[p^s_{(M,N)} - p^b_{(M,N)}] \leqslant (\gamma_b(N+1))^{-1}$.

(iii) *If* $M = N \geqslant 2$, *then* $E[p^s_{(M,N)} - p^b_{(M,N)}] \leqslant (\gamma_s(M+1))^{-1} + (\gamma_b(N+1))^{-1}$.

Now the expected gains from trade for a seller with valuation $\bar{s}$ when all M sellers and N buyers participate in the mechanism are given by

$$\underline{U}^s \equiv \begin{cases} \int_{\bar{s}}^{\bar{b}} (b - \bar{s})\, g_{M:N}(b)\, db & \text{if} \quad M \leqslant N \quad \text{and} \quad \bar{s} < \bar{b}, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $g_{M:N}$ is the density function of the Mth *highest* valuation out of N buyers' valuations. Likewise, the expected gains from trade for a buyer with valuation $\underline{b}$ when all M sellers and N buyers participate in the mechanism are given by

$$\underline{U} \equiv \begin{cases} \int_{\underline{s}}^{\underline{b}} (\underline{b} - s)\, f_{N:M(s)\, ds} & \text{if} \quad N \leqslant M \quad \text{and} \quad \underline{s} < \underline{b}, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where $f_{N:M}$ is the density function of the Nth *lowest* valuation out of M sellers' valuations. Therefore, as is shown in Williams [14], a necessary and sufficient condition for the mechanism to be efficient, ex-ante budget balancing, and interim individually rational is

$$E[k_{(M,N)}(p^s_{(M,N)} - p^b_{(M,N)})] \leqslant M\underline{U}^s + N\underline{U}^b. \tag{3}$$

Suppose condition (i) of Theorem 1 holds. Then, since $\underline{U}^s = 0$ by (1), and

$$\underline{U}^b = \int_{\underline{s}}^{\underline{b}} (\underline{b} - s)\, f_{N:M}(s)\, ds = \sum_{r=N}^{M} \binom{M}{r} \int_{\underline{s}}^{\underline{b}} [F(s)]^r\, [1 - F(s)]^{M-r}\, ds$$

[11] The proof is omitted since it can be derived straightforwardly from the definition of order statistics.

by (2), we have

$$E[k_{(M,N)}(p^s_{(M,N)} - p^b_{(M,N)})] \leqslant \frac{N}{\gamma_s(M+1)} \leqslant M\underline{U}^s + N\underline{U}^b$$

by Lemma 1. The second case of Theorem 1 can be similarly proved. We finally note that inequality (3) cannot be satisfied if $\underline{b} \leqslant \underline{s}$ and $\bar{b} \leqslant \bar{s}$, in particular if $[\underline{s}, \bar{s}] = [\underline{b}, \bar{b}]$.

Proof of Theorem 2. We first prove for the case when $M > N$. We set $\phi^s_i = 0$, $\forall i \in \mathscr{S}$. In addition, $\phi^b_j = \phi^b$, $\forall j \in \mathscr{B}$. In words, the market maker charges a uniform participation fee only to buyers. Then it is clear that all the sellers enter the mechanism, i.e., $S = \mathscr{S}$. Regarding who among the buyers will enter the mechanism with a strictly positive fee ϕ^b, we provide two lemmas below. First we have:

LEMMA 2. *Fix* $\mathscr{S}$ *with* $|\mathscr{S}| = M$, $\mathscr{B}$ *with* $|\mathscr{B}| = N$, *and* ϕ^b. *Then, for any buyer* $j \in \mathscr{B}$, *there exists a cut-off level of valuation, denoted by* $\beta_j(\phi^b; M, N)$, *such that* j *will enter the mechanism if and only if* $b_j \geqslant \beta_j(\phi^b; M, N)$.

Proof. We will first show that, for any given M, N, and ϕ^b, if b_j enters the mechanism then $\tilde{b}_j$ with $\tilde{b}_j > b_j$ will also enter. The reason is that the gain from participation is higher for $\tilde{b}_j$ than b_j regardless of the number n of participating buyers and the realization of other participants' valuations.

To see this, fix the number of participating buyers and also fix the valuations of the sellers and those of the participating buyers except j. Now when j's valuation is b_j, we can determine the volume of trade (k), the seller price (p^s), and the buyer price (p^b) by the rules of the mechanism. Let the corresponding quantities when j's valuation is $\tilde{b}_j$ be $\tilde{k}$, $\tilde{p}^s$, and $\tilde{p}^b$. We claim that the gain to $\bar{b}_j$ is no less that that to b_j. That is,

CLAIM 1. $(b_j - p^b)\, 1_{\{b_j \geqslant b_{(k)}\}} \leqslant (\tilde{b}_j - \tilde{p}^b)\, 1_{\{\tilde{b}_j \geqslant b_{(\tilde{k})}\}}$.

Proof. First observe that $b_j \geqslant b_{(k)}$ implies that $\tilde{b}_j \geqslant b_{(\tilde{k})}$. So, $1_{\{b_j \geqslant b_{(k)}\}} \leqslant 1_{\{\tilde{b}_j \geqslant b_{(\tilde{k})}\}}$. Next observe that $\tilde{p}^b$ is equal to p^b as long as $b_j \geqslant b_{(k)}$. (Note that $b_j \geqslant b_{(k)}$ implies that $k = \tilde{k}$. This, together with the fact that neither b_j nor $\tilde{b}_j$ can be $b_{(k+1)}$, implies that $\tilde{p}^b = \max\{s_{(\tilde{k})}, b_{(\tilde{k}+1)}\} = \max\{s_{(k)}, b_{(k+1)}\} = p^b$.) Since $\tilde{p}^b \leqslant \tilde{b}_j$ when $\tilde{b}_j \geqslant b_{(\tilde{k})}$, we proved the claim.

The expected gain from participation is an expectation over all possible n and all possible realizations of valuations, and thus we conclude that if b_j enters then $\tilde{b}_j$ with $b_j < \tilde{b}_j$ also enters.

Next we show that, for any M, N, and ϕ^b, the set of participating types (valuations) is indeed in the form of the (left-) closed interval

$[\beta_j(\phi^b; M, N), \bar{b}]$. This follows from the continuity of the expected gain with respect to the valuation b_j, and the usual assumption that j will participate if she is indifferent toward participation and non-participation. Proving continuity can be accomplished by proving the following claim.

CLAIM 2. *For b_j and $\tilde{b}_j$ with $b_j < \tilde{b}_j$, and for any n and any realization of other participants' valuations,*

$$(\tilde{b}_j - \tilde{p}^b)\, 1_{\{\tilde{b}_j \geqslant b_{(\tilde{k})}\}} - (b_j - p^b)\, 1_{\{b_j \geqslant b_{(k)}\}} \leqslant \tilde{b}_j - b_j. \tag{4}$$

Proof. If $b_j \geqslant b_{(k)}$, the *LHS* of (4) is equal to $\tilde{b}_j - b_j$. (See the proof of Claim 1.) Thus we are finished. Otherwise, i.e., when $b_j < b_{(k)}$, we have $\tilde{p}^b \geqslant b_j$ since $\tilde{p}^b \geqslant p^b \geqslant b_{(k+1)} \geqslant b_j$. Thus, the *LHS* of (4) is less than or equal to $\tilde{b}_j - b_j$. (It is $\tilde{b}_j - \tilde{p}^b$ when $\tilde{b}_j \geqslant b_{(\tilde{k})}$, and 0 otherwise.) This proves the claim. Therefore, we proved the lemma. ∎

Lemma 2 says that every buyer $j \in \mathscr{B}$ has a participation strategy $\beta_j(\phi^b; M, N)$ as a function of ϕ^b. We will assume that all the buyers use the same strategy, so that

$$\beta_j(\phi^b; M, N) = \beta(\phi^b; M, N) \qquad \text{for all} \quad j \in \mathscr{B}.$$

This amounts to assuming that buyers have the same expectation, that is, if two different buyers have the same valuation then they will form the same expectation regarding the behavior of others. Then we can derive an explicit formula for $\beta(\phi^b; M, N)$. Fix M and N, and think of a typical buyer j. The (gross) expected gain from participation for j (i) when her valuation is $b \in [\underline{b}, \bar{b}]$, and (ii) when other buyers $j' \neq j$ will participate if and only if $b_{j'} \geqslant b$, (that is, when b is the cut-off level of valuation) is given by

$$\begin{cases} \displaystyle\sum_{r=0}^{N-1} \binom{N-1}{r} [G(b)]^{N-1-r} [1-G(b)]^r \int_{\underline{s}}^{b} (b-s)\, f_{r+1:M}(s)\, ds \\ \qquad \text{if} \quad b \geqslant \underline{s}, \\ 0 \qquad \text{otherwise}, \end{cases} \tag{5}$$

where $f_{r+1:M}$ is the density function of the $(r+1)$-st order statistic from M sellers' valuations. Denote the above formula (5) by $\phi(b; M, N)$. Then we have established the relationship between ϕ^b and b, given by $\phi^b = \phi(b; M, N)$. The participation strategy $\beta(\phi^b; M, N)$ is nothing but the inverse of $\phi(b; M, N)$. Indeed, it is straightforward to see that $\phi(\cdot)$ is a strictly increasing function of b for given M and N, and thus has the strictly increasing inverse $\beta(\phi^b; M, N)$ in the range $\Phi = \{\phi^b : \phi^b = \phi(b, M, N)$ for some $\max\{\underline{s}, \underline{b}\} \leqslant b \leqslant \bar{b}\}$ of ϕ.[12] Note that Φ is an interval $[\underline{\phi}^b, \bar{\phi}^b]$ with $\underline{\phi}^b \geqslant 0$. Therefore, we can adequately modify $\beta(\phi^b; M, N)$ to be defined on $\bar{\mathbb{R}}_+$ as[13]

$$\beta(\phi^b; M, N) = \begin{cases} \underline{b} & \text{if } \phi^b \leqslant \underline{\phi}^b \\ \phi^{-1}(\phi^b; M, N) & \text{if } \phi^b \in (\underline{\phi}^b, \bar{\phi}^b] \\ \infty & \text{if } \phi^b > \bar{\phi}^b. \end{cases}$$

Summarizing the discussion, we have

LEMMA 3. *For any given M and N,*

(i) *the participation strategy $\beta(\phi^b; M, N)$ is an increasing functions of $\phi^b \in \mathbb{R}_+$, and is strictly increasing over Φ, and*

(ii) $\beta(\phi^b; M, N) = \underline{b}$ *when* $\phi^b = 0$.

The market maker's gross expected payment when there are n participating buyers is at most

$$nE[p^s_{(M,n)} - p^b_{(M,n)}] \leqslant n(\gamma_s(M+1))^{-1} \tag{6}$$

by Lemma 1(i). Therefore, for a participation fee $\phi^b = (\gamma_s(M+1))^{-1}$, ex-ante budget balance is satisfied. Let, for given M and N, $\hat{\phi}^b(M, N) = \inf\{\phi^b : \text{ex-ante budget balance is satisfied.}\}$ Obviously, $\hat{\phi}^b(M, N) \leqslant (\gamma_s(M+1))^{-1}$. Let $\hat{b}(M, N) = \beta(\hat{\phi}^b(M, N); M, N)$. The efficiency loss occurs whenever some buyers do not participate in the mechanism while there are non-selling sellers whose valuations are lower than those of the non-participating buyers. Let, for $1 \leqslant j \leqslant l \leqslant N$,

$$1(M, N; j, l) = 1_{\{(b_{(j)} < \hat{b}(M,N) \leqslant b_{(j-1)}) \,\&\, (s_{(l)} \leqslant b_{(l)},\, s_{(l+1)} > b_{(l+1)})\}}$$

be the indicator function which takes the value 1 when the actual realization of the traders' valuations is as on the right-hand side, and takes the value 0 otherwise. $1(M, N; j, l)$ indicates the situation where only $(j-1)$ buyers enter even though there are l mutually profitable trade possibilities. Then the expected efficiency loss is

$$I(M, N) = E\left[\sum_{j=1}^{N} \sum_{l=j}^{N} \sum_{r=j}^{l} (b_{(r)} - s_{(r)})\, 1(M, N; j, l)\right]. \tag{7}$$

[12] We suppress M and N in the definition of Φ for compactness. This also applies to $[\underline{\phi}_b, \bar{\phi}_b]$ below.
[13] It is obvious that buyers have nothing to lose from participation when $\phi^b = 0$.

Now

$$
\begin{aligned}
I(M, N) &\leqslant E\left[\sum_{j=1}^{N} \sum_{l=j}^{N} (l-j+1)(b_{(j)}-s_{(j)})\, 1(M, N; j, l)\right] \\
&\leqslant E\left[\sum_{j=1}^{N} (N-j+1)(b_{(j)}-s_{(j)})\, 1_{\{b_{(j)}<\hat{b}(M, N) \leqslant b_{(j-1)}\}}\right] \\
&\leqslant \sum_{r=0}^{N-1} \binom{N}{r} [G(\hat{b}(M, N))]^{N-r} [1-G(\hat{b}(M, N))]^{r} \\
&\quad \times (N-r) \int_{\underline{s}}^{\hat{b}(M, N)} (\hat{b}(M, N)-s)\, f_{r+1:M}(s)\, ds \\
&= G(\hat{b}(M, N))\, N \left\{\sum_{r=0}^{N-1} \binom{N-1}{r} [G(\hat{b}(M, N))]^{N-r-1}\right. \\
&\quad \left. \times [1-G(\hat{b}(M, N))]^{r} \int_{\underline{s}}^{\hat{b}(M, N)} (\hat{b}(M, N)-s)\, f_{r+1:M}(s)\, ds\right\} \\
&= G(\hat{b}(M, N))\, N\phi(\hat{b}(M, N); M, N) \qquad \text{by (5)} \\
&\leqslant G(\hat{b}(M, N))\, N/(\gamma_s(M+1)) \leqslant \frac{1}{\gamma_s}.
\end{aligned}
$$

This proves the case when $M>N$. The proof of the case when $M<N$ is symmetric. We set $\phi_j^b=0$, $\forall j \in \mathscr{B}$ and $\phi_i^s=\phi^s$, $\forall i \in \mathscr{S}$. Everything else follows in a symmetric way. When $M=N$, we can set $\phi_i^s=0$, $\forall i \in \mathscr{S}$ and $\phi_j^b=\phi^b$, $\forall j \in \mathscr{B}$. Then basically the same lines of reasoning as those in the first case go through, and we will get the desired bound $1/\gamma_s+1/\gamma_b$.[14]

REFERENCES

1. K. Chatterjee and W. Samuelson, Bargaining under incomplete information, *Oper. Res.* **31** (1983), 835–851.
2. E. Clarke, Multipart pricing of public goods, *Public Choice* **8** (1971), 19–33.
3. T. Gresik and M. Satterthwaite, The rate at which a simple market converges to efficiency as the number of trades increase: An asymptotic result for optimal trading mechanisms, *J. Econ. Theory* **48** (1989), 238–263.
4. T. Groves, Incentives in teams, *Econometrica* **41** (1973), 617–631.
5. K. Hagerty and W. Rogerson, Robust trading mechanisms, *J. Econ. Theory* **42** (1987), 94–107.

[14] Note that, in formula (6), the expectation must be taken conditional on the fact that the valuations of the participating buyers are at least $\beta(b; M, N)$. Still, it can be easily verified that $E[p^s_{(M,N)}-p^b_{(M,n)}] \leqslant (\gamma_s(M+1))^{-1}+(\gamma_b(N+1))^{-1}$.

6. W. Leninger, P. B. Linhart, and R. Radner, Equilibria of the sealed-bid mechanism for bargaining with incomplete information, *J. Econ. Theory* **48** (1989), 63–106.
7. L. Makowski and J. Ostroy, Vickrey–Clarke–Groves mechanisms and perfect competition, *J. Econ. Theory* **42** (1987), 244–261.
8. P. McAfee, A dominant strategy double auction, *J. Econ. Theory* **56** (1992), 434–450.
9. R. Myerson and M. Satterthwaite, Efficient mechanisms for bilateral trading, *J. Econ. Theory* **29** (1983), 265–281.
10. A. Rustichini, M. Satterthwaite, and S. Williams, Convergence to efficiency in a simple market with incomplete information, *Econometrica* **62** (1994), 1041–1063.
11. M. Satterthwaite and S. Williams, Bilateral trade with the sealed bid *k*-double auction: Existence and efficiency, *J. Econ. Theory* **48** (1989), 107–133.
12. M. Satterthwaite and S. Williams, The rate of convergence to efficiency in the buyer's bid double auction as the market becomes large, *Rev. Econ. Stud.* **56** (1989), 477–498.
13. W. Vickrey, Counterspeculation, auctions, and competitive sealed tenders, *J. Finance* **16** (1961), 8–37.
14. S. Williams, A characterization of efficient, bayesian incentive compatible mechanisms, *Econ. Theory* **14** (1999), 155–180.
15. R. Wilson, Incentive efficiency of double auctions, *Econometrica* **53** (1985), 1101–1115.