
Volume 47, issue 6, December 2011 ISSN 0304-4068
ELSEVIER
JOURNAL OF Mathematical ECONOMICS
Editor:
F. KÜBLER
Co-Editors:
A. CITANNA
A. KAJII
M. MACHINA
Advisory Board:
K.J. ARROW
R.J. AUMANN
T.F. BEWLEY
B. CORNET
J. GEANAKOPLOS
T. HENS
W. HILDENBRAND
E. MALINVAUD
A. MAS-COLELL
H. POLEMARCHAKIS
R. RADNER
H.E. SCARF
L.S. SHAPLEY
S. SMALE
H.F. SONNENSCHEIN
Editorial Board:
Associate Editors:
R. McLEAN
J. SAGI
B.E. ALLEN
N.I. AL-NAJJAR
S. GRANT
C. HARA
J-J. HERINGS
J. HORNER
T.J. KEHOE
T. KEISTER
P. MONTEIRO KLINGER
M.J.P. MAGILL
E. MINELLI
J. QUAH
F.H. RIEDEL
A. RUSTICHIN
K. SCHMEDDERS
M.K. SCHNEIDER
C. SHANNON
P. SICONOLFI
W. TROCKEL
N. WANG
J. WERNER
U. ÜNVER
N. YANNELIS
W. ZAME
CONTENTS
J. Honda, Noise-independent selection in global games and monotone potential maximizer: A symmetric 3 × 3 example 663
K. Yoon, Optimal mechanism design when both allocative inefficiency and expenditure inefficiency matter 670
K.-I. Akao, T. Kamihigashi and K. Nishimura, Monotonicity and continuity of the critical capital stock in the Dechert–Nishimura model 677
D. Oyama and S. Takahashi, On the relationship between robustness to incomplete information and noise-independent selection in global games 683
B.K. Beare, Measure preserving derivatives and the pricing kernel puzzle 689
A. Lachapelle and F. Santambrogio, On the strategic use of risk and undesirable goods in multidimensional screening 698
C.S. Webb and H. Zank, Accounting for optimism and pessimism in expected utility 706
O. Lev, A two-dimensional problem of revenue maximization 718
H. Uno, Strategic complementarities and nested potential games 728
E. Tsakas and M. Voorneveld, On consensus through communication without a commonly known protocol 733
(Contents continued on Outside Back Cover)
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Journal of Mathematical Economics 47 (2011) 670–676

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Optimal mechanism design when both allocative inefficiency and expenditure inefficiency matter

Kiho Yoon *

Department of Economics, Korea University, Anam-dong, Sungbuk-gu, Seoul, 136-701, Republic of Korea

ARTICLE INFO

Article history:
Received 10 September 2010
Received in revised form
30 August 2011
Accepted 11 September 2011
Available online 18 September 2011

Keywords:
Mechanism
Random assignment
Winner-take-all assignment
Rank-order rules
Expenditure inefficiency

ABSTRACT

We characterize the structure of optimal assignment rules when both allocative inefficiency and expenditure inefficiency (e.g., rent-seeking) are present. We find that the optimal structure critically depends on how the hazard rate of the value distribution behaves, and that it is often optimal to use probabilistic assignment rules so that the winner of the object is not always the one with the highest valuation. We also find that the inefficiency of the optimal assignment rule decreases as the variability of the value distribution increases.

1. Introduction

People spend valuable resources to obtain what they desire, and this expenditure is not always efficiently invested. Education is probably one of the most famous examples after Spence (1973): over-investments or even completely wasteful investments in signaling occur to obtain diplomas. Rent-seeking is another famous example after Tullock (1980): rents are dissipated due to wasteful behaviors. In these situations, two sources of inefficiency may occur. The first is inefficiency from the (at least partial) waste of valuable resources. The second is allocative inefficiency that may result if the objects of interest are not assigned to those who value them most or who can utilize them most productively.

A simple example will help to clarify the main theme of this paper.[1] There is one object to be assigned to either one of two players. Assume first that both players attribute a value of one dollar to the object, and that this fact is common knowledge. If the object is assigned to the player who throws more pennies into a big pond,[2] each player will waste 50 cents in expectation. The total inefficiency is one dollar, and the rent is fully dissipated. On the other hand, if the object is assigned randomly, no inefficiency results. Hence, random assignment performs better. Assume next that one of the players attributes a value of one dollar, while the other attributes zero to the object. Players know their respective valuations, but others do not know whose valuation is one. Then, the player with a low valuation throws no penny obviously, while the player with a high valuation throws one penny to get the object. This leads to (almost) no inefficiency. On the other hand, if the object is assigned randomly then the player with zero value will get the object with probability 0.5, resulting in an allocative inefficiency of 50 cents. In this case, the money-throwing method performs better.

This example shows that, faced with two kinds of inefficiency, allocative inefficiency and expenditure inefficiency, uncertainty regarding true valuations matters for the relative performance of alternative assignment rules. There exists no value uncertainty in the first case, thus no inefficiency from mis-allocation occurs. All we need to take care is the inefficiency from wasteful expenditures. Value uncertainty is significant in the second case, and we have to consider both expenditure inefficiency and allocative inefficiency.

We approach this problem from a mechanism design perspective under incomplete information. Hence, we assume that players have private information regarding their valuations for the object, and look for optimal assignment rules that minimize the sum of allocative inefficiency and expenditure inefficiency. In particular, we focus on the *rank-order* assignment rules that respect the order but not the exact level of players' expenditures. The rank-order rule, in which only relative (as opposed to absolute) performance matters, is widely used in practice: elections, sports competitions,

* Tel.: +82 2 3290 2222; fax: +82 2 928 4948.
E-mail address: kiho@korea.ac.kr.
URL: http://econ.korea.ac.kr/~kiho.

[1] Though this example postulates complete information for illustrative purpose, this paper in fact studies the incomplete information setting.

[2] This is an all-pay auction under complete information, for which $F(x) = x$ for $x \in [0, 1]$ is the cumulative distribution function of the unique equilibrium strategy.

0304-4068/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.jmateco.2011.09.002

promotion in organizations, and classroom grading are just a few examples. It is a simple scheme to implement, especially when individual performance is hard to measure on a cardinal scale. Moreover, as the works including Lazear and Rosen (1981) and Green and Stokey (1983) show, it may in fact be an optimal arrangement for many economic situations.[3]

We find that the optimal structure of assignment rules depends critically on how the hazard rate of the value distribution behaves, and that it is often optimal to use probabilistic assignment rules in which the winner of the object is not always the one with the highest valuation. We also find that the inefficiency of the optimal assignment rule decreases as the variability of the value distribution increases. These results are obtained in a clear and straightforward fashion by applying the theory of stochastic orders. This theory of stochastic orders among differences of order statistics, pioneered by Barlow and Proschan (1966), was recently introduced to various models in economics by Moldovanu et al. (2007, 2008) and Hoppe et al. (2009). In particular, the present paper is technically similar to the last paper that studies the assortative two-sided matching with costly signals to see how the change in either side of the matching affects the relevant variables such as the welfare and the signaling efforts. In comparison, we study the optimal choice of assignment rules depending on the distribution of one population.[4]

The problem of this paper may also be analyzed by the optimal mechanism design approach initiated by Myerson (1981). Then, it is possible to consider more general assignment rules, beyond rank-order assignment rules. Since this approach is quite well-established in the literature, however, we relegate the discussion on the general mechanism design approach to the Appendix. The analysis in the Appendix shows that our results (in particular, Propositions 2 and 3) continue to hold even when we consider more general assignment rules. Moreover, the theory of stochastic orders enables us to obtain a meaningful comparative static result with respect to the changes in the value distribution (Proposition 4).

Hartline and Roughgarden (2008), Chakravarty and Kaplan (2009) and Condorelli (2011) adopted the optimal mechanism design approach to study similar problems. The last paper, in particular, derives similar results to Proposition 2. Compared to Condorelli (2011), the current paper encompasses the case when only parts of the expenditure are counted as inefficiency as well as establishes a result as to how the variability of the value distribution affects efficiency. On the other hand, Condorelli discusses heterogeneous objects case and then the implementation via priority lists and queues. Also related are Suen (1989), Taylor et al. (2003), and Koh et al. (2006) that compare the waiting-line auction and the lottery for specific distributions.[5]

2. Main results

2.1. The model

We consider a situation where one object is to be assigned to one of the players in $N = \{1, \ldots, n\}$.[6] The object may be tangible such as a product or a specific position; or it may be intangible such as a government contract, political favor, or social status. It may also be a mating partner in the case of biological contests. Each player $i \in N$ has a valuation v_i for the object. We postulate incomplete information so that player i knows his valuation v_i, while others only know its distribution. We assume that each player's valuation is drawn independently from the interval $[\underline{v}, \overline{v}]$ with $0 \leq \underline{v} \leq \overline{v} \leq \infty$ according to a common distribution F.[7] We assume further that F admits a continuous density function f which is strictly positive on the interval $[\underline{v}, \overline{v}]$.

Player i exerts an observable expenditure $x_i \in \mathbb{R}_+$ to win the object. This expenditure is assumed to be an *unconditional commitment* of resources.[8] That is, each player exerts x_i whether or not he actually gets the object. This expenditure may be a monetary bid as in all-pay auctions, an effort level in contests, time in a waiting line, or a costly investment as in a signaling context or in a biological context.

A mechanism is an assignment rule $p = (p_1, \ldots, p_n)$ that depends on the expenditure vector $x = (x_1, \ldots, x_n)$, where p_i is the probability that player i gets the object. To be feasible, the assignment rule must satisfy $p_i(x) \geq 0$ for all $i \in N$ and $\sum_{j=1}^{n} p_j(x) = 1$, for all $x \in \mathbb{R}_+^n$. Player i's payoff is $u_i \equiv p_i v_i - x_i$ when his valuation is v_i and he exerts an expenditure of x_i. We want to note that the linear disutility of expenditure is not as restricted as it appears. We may alternatively introduce a general cost function $c(x_i, v_i)$ so that $u_i \equiv p_i v_i - c(x_i, v_i)$, and assume that $c(x_i, v_i)$ is strictly increasing in x_i. This function can be either convex or concave in x_i as well as either increasing or decreasing in v_i. It is an easy and interesting exercise to observe that virtually the same results hold in the following if we replace x_i with $c(x_i, v_i)$, in particular, in the expenditure inefficiency and the net efficiency defined below.

Each player chooses his expenditure to maximize the expected payoff, given other players' expenditures and the assignment rule. Hence, player j's strategy in the mechanism is a function $e_j : [\underline{v}, \overline{v}] \rightarrow \mathbb{R}_+$ that maps his valuation to the expenditure level. Given others' strategies, player i's problem with valuation v_i is

$$\max_{x_i} E_{v_{-i}}[p_i(x_i, e_{-i}(v_{-i}))v_i - x_i].$$

In the above expression, we follow the convention that the subscript $-i$ pertains to players other than player i. For example, $v_{-i} = (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

It will be convenient to work with order statistics. Hence, let $v_{1:n} \geq v_{2:n} \geq \cdots \geq v_{n:n}$ be the order statistics of $v_1, \ldots, v_n$. Note that $v_{k:n}$ is the k-th highest among n valuations drawn from the common distribution F. The distribution and the density of $v_{k:n}$ are denoted by $F_{k:n}$ and $f_{k:n}$, respectively.[9] We will also deal with players' valuations except player i's, so we can similarly have the order statistic $v_{k:n-1}$ and the corresponding functions $F_{k:n-1}$ and $f_{k:n-1}$ of the k-th highest among $(n-1)$ valuations.

As discussed in the introduction, we restrict our attention to the class of assignment rules that respect the order, but not the amounts of expenditure. That is, given a vector $(x_1, \ldots, x_n)$ of players' expenditures, a probability π_1 of win is given to the player with the highest expenditure $x_{1:n}$, a probability π_2 is given to the player with the second highest expenditure $x_{2:n}$, and so on.[10] Let

---

3 Frankel (2010) recently shows that the rank-order rule may be an optimal mechanism under certain circumstances if the worst-case optimality criterion instead of Bayesian optimality criterion is considered.

4 In a sense, this paper takes one of the two sides in Hoppe et al. (2009) as a decision variable and chooses optimally according to the changes in the remaining side.

5 The main analysis of Taylor et al. (2003) deals only with the Beta distribution numerically, while Koh et al. (2006) consider 4 specific (power, Weibull, logistic, and Beta) distributions.

6 It is a straightforward matter to extend the analysis to the case of multiple homogeneous objects. See the discussion at the end of this section.

7 The Appendix contains an analysis of the asymmetric player case.

8 As far as the author is aware, Amman and Leininger (1995, 1996) were the first to make the distinction between unconditional commitment and conditional commitment.

9 $F_{k:n}(z) = \sum_{r=0}^{k-1} \binom{n}{r} F(z)^{n-r}[1 - F(z)]^r$ and $f_{k:n}(z) = \frac{n!}{(k-1)!(n-k)!} F(z)^{n-k}[1 - F(z)]^{k-1} f(z)$.

10 Ties can be dealt with by combining the relevant probabilities and assigning equal chances.

us call these assignment rules the *rank-order assignment rules*. We will henceforth denote a mechanism by the rank-order assignment rule $(\pi_1, \ldots, \pi_n)$, with π_k being the probability that a player whose expenditure is the k-th highest wins the object.

2.2. Analysis

Since players are symmetric, we begin with a heuristic derivation of symmetric equilibrium strategies. Suppose that players other than i follow a symmetric, increasing and differentiable equilibrium strategy $e(\cdot)$. First, it is straightforward to see that player i will never optimally exert an expenditure $x_i > e(\overline{v})$. Second, it is also easy to see that a player with valuation $\underline{v}$ will optimally choose an expenditure of zero. Then player i's expected payoff when his valuation is v_i and he exerts an expenditure of $e(w_i)$ is

$$\begin{aligned} U_i(v_i; w_i) &= v_i\big\{\pi_1 F_{1:n-1}(w_i) + \pi_2[F_{2:n-1}(w_i) - F_{1:n-1}(w_i)] \\ &\quad + \cdots + \pi_{n-1}[F_{n-1:n-1}(w_i) - F_{n-2:n-1}(w_i)] \\ &\quad + \pi_n[1 - F_{n-1:n-1}(w_i)]\big\} - e(w_i) \\ &= v_i\left\{\sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})F_{k:n-1}(w_i) + \pi_n\right\} - e(w_i). \end{aligned}$$

In words, if player i has a true valuation of v_i but exerts an expenditure as if his valuation is w_i, he will get the object with probability π_1 when his expenditure is the highest, i.e., $v_{1:n-1} \le w_i$, he will get the object with probability π_2 when his expenditure is the second highest, i.e., $v_{2:n-1} \le w_i < v_{1:n-1}, \ldots,$ and he will get the object with probability π_n when his expenditure exceeds no other players' expenditures, i.e., $w < v_{n-1:n-1}$. Note well that $F_{k:n-1}(w_i) - F_{k-1:n-1}(w_i)$ is the probability that w_i lies between the $(k-1)$-th and the k-th highest among $(n-1)$ other players' valuations, i.e., $Prob[v_{k:n-1} \le w_i < v_{k-1:n-1}]$.

The first-order condition for the payoff maximization with respect to w_i is

$$v_i\sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})f_{k:n-1}(w_i) = e'(w_i).$$

Since $w_i = v_i$ at a symmetric equilibrium, we have the differential equation

$$e'(v_i) = v_i\sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})f_{k:n-1}(v_i),$$

from which we get the equilibrium strategy

$$e(v_i) = \sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})\int_{\underline{v}}^{v_i} w\,\mathrm{d}F_{k:n-1}(w).$$

While this is only a heuristic derivation, the following proposition shows that this is indeed an equilibrium.

Proposition 1. *Suppose that the rank-order assignment rule $(\pi_1, \ldots, \pi_n)$ satisfies $\pi_1 \ge \cdots \ge \pi_n$. Then, the symmetric equilibrium strategy of the mechanism is*

$$e(v_i) = \sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})\int_{\underline{v}}^{v_i} w\,\mathrm{d}F_{k:n-1}(w)$$

for all $i \in N$ and $v_i \in [\underline{v}, \overline{v}]$.

Proof. If $\pi_1 = \pi_2 = \cdots = \pi_n$, then the equilibrium strategy is $e(v_i) = 0$ for all $v_i \in [\underline{v}, \overline{v}]$. Now, if $\pi_1 \ge \cdots \ge \pi_n$ with at least one strict inequality, then $e(v_i)$ is strictly increasing and continuous. We next show that $U_i(v_i; w_i)$ is maximized by choosing $w_i = v_i$. We have

$$\begin{aligned} &U_i(v_i; v_i) - U_i(v_i; w_i) \\ &= \sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})\left\{v_i[F_{k:n-1}(v_i) - F_{k:n-1}(w_i)] - \int_{w_i}^{v_i} w\,\mathrm{d}F_{k:n-1}(w)\right\} \\ &= \sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})\left\{\int_{w_i}^{v_i} F_{k:n-1}(w)\,\mathrm{d}w - (v_i - w_i)F_{k:n-1}(w_i)\right\}, \end{aligned}$$

where the second equality follows from integration by parts. Observe that the last term is always nonnegative regardless of whether $w_i \le v_i$ or $w_i \ge v_i$. □

We will henceforth assume that the rank-order assignment rule satisfies monotonicity, i.e., $\pi_1 \ge \cdots \ge \pi_n$. We note that this is with no loss of generality since the general mechanism design approach in the Appendix shows that monotonicity must be satisfied in any incentive compatible assignment rule.[11] The assignment rule is called the winner-take-all assignment when $\pi_1 = 1$ and $\pi_k = 0$ for $k = 2, \ldots, n$. It is called the random assignment when $\pi_1 = \cdots = \pi_n = \frac{1}{n}$.

Player i's expected expenditure is

$$\begin{aligned} E[e(v_i)] &= \int_{\underline{v}}^{\overline{v}}\sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})\int_{\underline{v}}^{v_i} w\,\mathrm{d}F_{k:n-1}(w)\,\mathrm{d}F(v_i) \\ &= \sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})\left\{\int_{\underline{v}}^{\overline{v}} w\,\mathrm{d}F_{k:n-1}(w) - \frac{n-k}{n}\int_{\underline{v}}^{\overline{v}} w\,\mathrm{d}F_{k:n}(w)\right\} \\ &= \frac{1}{n}\sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})\{n\mu_{k:n-1} - (n-k)\mu_{k:n}\} \\ &= \frac{1}{n}\sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})k\mu_{k+1:n} \end{aligned}$$

where $\mu_{k:n}$ is the expected value of the k-th highest among n valuations, and the last equality follows from a well-known recurrence relation.[12] Hence, the total expenditure is $nE[e(v_i)] = \sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})k\mu_{k+1:n}$.

We are interested in designing an optimal mechanism when the (excess) expenditure is viewed as socially wasteful investment. Thus, we can imagine a social welfare loss function that is increasing in the total expenditure $\sum_{i=1}^{n} x_i$. It may be identically zero if the expenditures are purely monetary transfers, as in the case of all-pay auctions. It may be equal to the total expenditure, as in the case of wasteful rent-seeking behavior. Generally, however, real resources are expended up to more than the desirable level as we know well from the signaling literature or the contest literature. We will call the social welfare loss due to wasteful expenditure as *expenditure inefficiency*, to distinguish it from the more conventional inefficiency due to mis-allocation, i.e., *allocative inefficiency*. We take expenditure inefficiency as a fraction of the total expenditure, so that it is

$$\alpha\sum_{k=1}^{n-1}(\pi_k - \pi_{k+1})k\mu_{k+1:n}$$

11 We also note that, as the analysis in the Appendix implies, there cannot exist asymmetric equilibria that may achieve higher net efficiency (to be defined shortly) when players are symmetric. Therefore, it is with no loss of generality to restrict our attention to the symmetric equilibrium.

12 See David (1981) for a good introduction to order statistics.

with $0 \leq \alpha \leq 1$. Note that expenditure inefficiency is non-negative for any assignment rule, and is zero when $\alpha = 0$ or the assignment rule is the random assignment.

We now turn to allocative inefficiency. Given the increasing equilibrium expenditure function, the allocative performance of an assignment rule $(\pi_1, \ldots, \pi_n)$ is

$$E[\pi_1 v_{1:n} + \cdots + \pi_n v_{n:n}].$$

Observe that the maximum allocative efficiency occurs when the object is always assigned to the player with the highest valuation, which is $E[v_{1:n}]$. Hence, allocative inefficiency is

$$\mu_{1:n} - \sum_{k=1}^{n} \pi_k \mu_{k:n}.$$

There is no allocative inefficiency under the winner-take-all assignment. On the other hand, allocative inefficiency is $\mu_{1:n} - (1/n)\sum_{k=1}^{n} \mu_{k:n} = \mu_{1:n} - \mu_{1:1}$ under the random assignment.[13] Observe also that allocative inefficiency decreases monotonically as more mass is put on higher π_k's, since $\mu_{k:n} > \mu_{j:n}$ for $k < j$. We also note that allocative inefficiency is non-negative for any assignment rule since $\mu_{1:n} \geq \cdots \geq \mu_{n:n}$.

The *net efficiency* of an assignment rule may be defined as its allocative performance net of expenditure inefficiency. This is

$$\begin{aligned} &\sum_{k=1}^{n} \pi_k \mu_{k:n} - \alpha \sum_{k=1}^{n-1} (\pi_k - \pi_{k+1}) k \mu_{k+1:n} \\ &= \sum_{k=1}^{n} [\pi_k - \alpha(k-1)(\pi_{k-1} - \pi_k)] \mu_{k:n} \\ &= \sum_{k=1}^{n} [\alpha k(\mu_{k:n} - \mu_{k+1:n}) + (1-\alpha)\mu_{k:n}] \pi_k \qquad (1) \end{aligned}$$

with the convention that $\mu_{n+1:n} = 0$. When $\alpha = 1$, net inefficiency becomes

$$\sum_{k=1}^{n} [k\pi_k - (k-1)\pi_{k-1}] \mu_{k:n} = \sum_{k=1}^{n} k(\mu_{k:n} - \mu_{k+1:n}) \pi_k. \qquad (2)$$

A mechanism that maximizes net efficiency is called an optimal assignment rule.

We are ready to show our main results. Before doing so, we collect some useful facts from the theory of stochastic orders: (i) A random variable X or its distribution F is said to be IFR if the hazard rate (i.e., the failure rate) $F'(x)/(1 - F(x))$ is increasing, and DFR if the hazard rate is decreasing. Equivalently, it is IFR (DFR) if the survival rate $1 - F(x)$ is log-concave (log-convex). Examples of IFR distributions are exponential, uniform, normal, logistic, power (for $c \geq 1$), Weibull (for $c \geq 1$), and gamma (for $c \geq 1$), while those of DFR distributions are exponential, Weibull (for $0 < c \leq 1$), gamma (for $0 < c \leq 1$), and Pareto.[14,15] (ii) A random variable X with distribution F is said to be stochastically smaller than another random variable Y with distribution G if $F(z) \geq G(z)$ for all z. This relationship is denoted by $X \leq_{st} Y$. It is often said that Y (first-order) stochastically dominates X. (iii) For the *normalized spacing* $k(X_{k:n} - X_{k+1:n})$, the following lemma is useful for us.

Lemma 1. *Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed IFR (DFR) random variables. Then*

$$k(X_{k:n} - X_{k+1:n}) \geq_{st} (\leq_{st}) (k-1)(X_{k-1:n} - X_{k:n})$$

for $k = 2, \ldots, n-1$.

This lemma is quite well-known in the theory of stochastic orders.[16] Observe that the expression (1) for net efficiency is a linear function of π_k's with coefficients $[\alpha k(\mu_{k:n} - \mu_{k+1:n}) + (1-\alpha)\mu_{k:n}]$, where $k(\mu_{k:n} - \mu_{k+1:n})$ is the expected value of the normalized spacing.

It is obvious to see that the winner-take-all assignment is optimal when $\alpha = 0$, i.e., without concern for the expenditure. We start with the opposite case of $\alpha = 1$.

Proposition 2. *Assume $\alpha = 1$. When the distribution F is IFR, the random assignment, i.e., $\pi_1 = \cdots = \pi_n = 1/n$, is an optimal assignment. On the other hand, when the distribution F is DFR, the winner-take-all assignment, i.e., $\pi_1 = 1$, is an optimal assignment.*

Proof. By Lemma 1, the coefficients of expression (2) are increasing (decreasing, resp.) when F is IFR (DFR, resp.). Hence, the results follow by solving the simple linear programming problem of maximizing $\sum_{k=1}^{n} k(\mu_{k:n} - \mu_{k+1:n})\pi_k$ with respect to π_k's subject to $\pi_1 \geq \cdots \geq \pi_n \geq 0$ and $\pi_1 + \cdots + \pi_n = 1$. □

This result may be interpreted as follows. Observe first that players with high valuations are relatively abundant (scarce, resp.) under IFR (DFR, resp.). Intuitively, the relative scarcity of high valuations may induce (i) the appreciation of high valuations for the attainment of allocative efficiency and (ii) less competition (expenditure inefficiency) among those with high valuations. Thus, allocative performance weighs more than expenditure inefficiency. This makes the winner-take-all assignment optimal under DFR. The opposite holds true under IFR, making the random assignment optimal. As a matter of fact, random assignments (or alternatively called as lotteries) are widely used in many allocation problems. While fairness is often invoked as a major concern, this paper shows that efficiency concern may lead to the use of random assignment rules when expenditure inefficiency is incorporated.[17] We next turn to the general case.

Proposition 3. *Assume $0 \leq \alpha \leq 1$. The winner-take-all assignment, i.e., $\pi_1 = 1$, is an optimal assignment when the distribution F is DFR. On the other hand, any assignment rule may be optimal depending on the relative importance of expenditure inefficiency, i.e., on the magnitude of α, when F is IFR.*

Proof. Observe in expression (1) that net efficiency is a convex combination of the expected value of k-th normalized spacing and the expected value of the k-th order statistic. Since the former is decreasing by Lemma 1 when F is DFR and the latter is decreasing also, the first assertion follows. As for the second assertion, we give an example below. □

Example 1 (*The Optimal Assignment may Depend on α when F is IFR*)**.** Let $n = 2$ so that there are two players, and let $F(x) = x$ so that the distribution is uniform on the interval $[0, 1]$. Net efficiency is given by

$$\frac{1}{3}(1 + \alpha + (1 - 2\alpha)\pi_1).$$

Hence, the winner-take-all assignment is optimal when $\alpha < 1/2$ and the random assignment is optimal when $\alpha > 1/2$. Any assignment $\pi_1 \in [0, 1]$ with $\pi_2 = 1 - \pi_1$ is optimal when $\alpha = 1/2$.

13 Note that $E[v_{1:n} + \cdots + v_{n:n}] = E[v_1 + \cdots + v_n] = nE[v_1]$.

14 Here, c is the shape parameter. We discuss some of these distributions in more detail below.

15 These are well-known standard facts. For an interesting discussion of log-concavity and log-convexity in the economics literature, see Bagnoli and Bergstrom (2005) and the references therein.

16 Shaked and Shanthikumar (2007) is a very good reference.

17 It is worthwhile to note at this juncture that Budish et al. (2011) recently focus on, among others, implementing random allocation mechanisms when a set of feasibility constraints are imposed. They especially pay attention to two alternative mechanisms proposed by Hylland and Zeckhauser (1979) and Bogomolnaia and Moulin (2001).

Note that, given the nature of our linear programming, either the winner-take-all assignment or the random assignment will be generically an optimal rank-order assignment when there are only two players. We may get other optimal rank-order assignment rules when $n \geq 3$.

Example 2 (*The Optimal Assignment may be Intermediate when F is neither IFR nor DFR*)**.** Let $n = 3$ and $F(x) = \sqrt{x}$ on the interval [0, 1]. Observe that this distribution is neither IFR nor DFR. Since $(\mu_{1:3}, \mu_{2:3}, \mu_{3,3}) = (6/10, 3/10, 1/10)$ and net efficiency is given by

$$(\mu_{1:3} - \mu_{2:3})\pi_1 + 2(\mu_{2:3} - \mu_{3:3})\pi_2 + 3\mu_{3:3}\pi_3$$

when $\alpha = 1$, the optimal assignment rule is $\pi_1 = \pi_2 = 1/2$ and $\pi_3 = 0$.

While we have obtained these results for rank-order assignment rules, we want to emphasize that Propositions 2 and 3 are still valid even when we consider more general assignment rules. That is, for the distributions in Propositions 2 and 3, these assignment rules are optimal mechanisms not just among rank-order assignment rules but among all possible assignment rules.[18] Please refer the Appendix for details as well as for a discussion on how the restriction to rank-order assignment rules may be binding.

2.3. Comparative statics

Intuition may suggest that the optimal assignment becomes more like the winner-take-all assignment than the random assignment as the uncertainty of valuation increases, i.e., the variability or the dispersion of the distribution increases. The following simple counter-example shows that this is not true. The gamma distribution with parameter c has the density of $f(x) = x^{c-1}e^{-x}/\Gamma(c)$, with both the mean and the variance being equal to c. Since the variance increases as c increases, we may think that the gamma distribution with $c = 2$ exhibits greater uncertainty than the gamma distribution with $c = 1/2$. But it follows from Proposition 2 that the optimal assignment when $\alpha = 1$ is the random assignment for the former, while it is the winner-take-all assignment for the latter since the gamma distribution is IFR for $c \geq 1$ and DFR for $c \leq 1$.[19]

Nevertheless, we now show that the net efficiency of the optimal assignment rule for any $\alpha \in [0, 1]$ increases as the variability of the distribution increases. We need additional facts from the theory of stochastic orders. First, a vector $c = (c_1, \ldots, c_n)$ is said to be smaller in the majorization order than the vector $d = (d_1, \ldots, d_n)$, denoted $c \prec d$, if $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} d_i$ and if $\sum_{i=1}^{j} c_{[i]} \leq \sum_{i=1}^{j} d_{[i]}$ for $j = 1, \ldots, n-1$, where $c_{[i]}$ and $d_{[i]}$ are the i-th largest elements of c and d, respectively, for $i = 1, \ldots, n$. Now, let X and Y be two random variables defined on the intervals $[a, b]$ and $[a', b']$ respectively, with distributions F and G which admit strictly positive densities f and g. The following lemma is from Barlow and Proschan (1966).[20,21]

Lemma 2. *Let $a = a' = 0$, $F(0) = G(0) = 0$, $G^{-1}F(x)/x$ be increasing on $[0, b]$, and $E[X] = E[Y]$.*[22] *Then,*

(i) $(\mu^X_{1:n}, \ldots, \mu^X_{n:n}) \prec (\mu^Y_{1:n}, \ldots, \mu^Y_{n:n})$ *in the majorization order.*
(ii) $\sum_{k=1}^{n} c_k k(\mu^X_{k:n} - \mu^X_{k+1:n}) \leq \sum_{k=1}^{n} c_k k(\mu^Y_{k:n} - \mu^Y_{k+1:n})$ *for* $c_1 \geq c_2 \geq \cdots \geq c_n$.

In the lemma, $\mu^X_{k:n}$ and $\mu^Y_{k:n}$ are respectively the expected values of the k-th highest among n order statistics independently drawn from the distributions F and G. It is known that if $G^{-1}F(x)/x$ is increasing and $E[X] = E[Y]$ then X second-order stochastically dominates Y, i.e., $\int_0^z F(t)\,dt \leq \int_0^z G(t)\,dt$ for all z.[23] Thus, Y is more variable than X. We have:

Proposition 4. *When the distribution of valuations becomes more variable in the sense that it changes from F to G with $G^{-1}F(x)/x$ increasing, together with the assumptions of Lemma 2, both allocative performance and net efficiency of the optimal assignment rule increase.*

Proof. First, it is easy to see that allocative performance $\sum_{k=1}^{n} \pi_k \mu_{k:n}$ is higher under G for any $\pi_1 \geq \cdots \geq \pi_n$ since $(\mu^X_{1:n}, \ldots, \mu^X_{n:n}) \prec (\mu^Y_{1:n}, \ldots, \mu^Y_{n:n})$ in the majorization order by Lemma 2(i). Next, since net efficiency in Eq. (1) is

$$\alpha \sum_{k=1}^{n} k(\mu_{k:n} - \mu_{k+1:n})\pi_k + (1-\alpha)\sum_{k=1}^{n} \mu_{k:n}\pi_k$$

and both summations increase under G by Lemma 2, we conclude that the maximum net efficiency under G must increase. □

Hence, allocative efficiency dominates expenditure inefficiency as the variability of the distribution increases. In other words, even if expenditure inefficiency may increase as the variability increases, net efficiency unambiguously improves for any portion α that is considered as inefficiency.

We next see the effect of changes in the number of players. Suppose the number of players increases from n to n', with $n' > n$. For any given assignment rule $(\pi_1, \ldots, \pi_n)$, let $(\pi_1, \ldots, \pi_{n'})$ to be $(\pi_1, \ldots, \pi_n, 0, \ldots, 0)$, i.e., set $\pi_{n+1} = \cdots = \pi_{n'} = 0$. Then, it is easy to see that allocative performance $\sum_{k=1}^{n} \pi_k \mu_{k:n}$ increases for any $(\pi_1, \ldots, \pi_n)$ since $\mu_{k:n}$ is an increasing function of n for all $k = 1, \ldots, n$.[24] As for net efficiency, assume first that the distribution F is DFR. The winner-take-all assignment is an optimal assignment by Proposition 3, so net efficiency is $\alpha(\mu_{1:n} - \mu_{2:n}) + (1-\alpha)\mu_{1:n}$. It increases in n since $\mu_{1:n} - \mu_{2:n}$ as well as $\mu_{1:n}$ increases as n increases.[25] Thus, more competition leads to higher net efficiency even when expenditure inefficiency is considered. On the other hand, net efficiency does not change when the random assignment is an optimal assignment rule, such as when F is IFR and $\alpha = 1$, since allocative performance is $E[v_1]$ and expenditure inefficiency is zero.

We finally note that the analysis can be extended in a straightforward manner to the case of multiple objects. All we need to modify is the feasibility of the assignment rule. Hence, if m homogeneous and indivisible objects are to be assigned to n players each of whom demands at most one unit with $m < n$, then the rank-order assignment rule must satisfy $0 \leq \pi_k \leq 1$ for all $k = 1, \ldots, n$ and $\sum_{i=1}^{n} \pi_k = m$. Then, players' equilibrium strategies remain the same, and all the results are essentially intact. All we need to do is to adjust properly according to the feasibility of assignment rules: For example, an optimal assignment rule is $\pi_1 = \pi_2 = \cdots = \pi_m = 1$ when F is DFR, while it is $\pi_1 = \pi_2 = \cdots = \pi_n = \frac{m}{n}$ when F is IFR and $\alpha = 1$.

---

18 To be precise, we need to set $\underline{v} = 0$.

19 For many characterizations of useful distributions, see Patel et al. (1976).

20 This is a restatement of parts of their Theorem 3.9. Note that the order statistics are arranged in a decreasing order in this paper, while they are arranged in an increasing order in Barlow and Proschan (1966).

21 As noted in the introduction, Hoppe et al. (2009) is one of the first papers that utilize this important result in economics.

22 $G^{-1}F(x)$ is said to be starshaped in x when $G^{-1}F(x)/x$ is increasing in $x \geq 0$.

23 See Theorem 4.B.4. of Shaked and Shanthikumar (2007).

24 $\mu_{k:n}$ is an increasing function of n since $v_{k:n} \leq_{lr} v_{k:n'}$, i.e., the former is smaller than the latter in the likelihood ratio order for $n' > n$.

25 $\mu_{1:n} - \mu_{2:n}$ increases in n since $v_{k:n} - v_{k+1:n} \leq_{hr} v_{k:n'} - v_{k+1:n'}$, i.e., the former is smaller than the latter in the hazard rate order for $n' > n$ when the distribution F is DFR and absolutely continuous. See, for example, Theorem 1.B.31 of Shaked and Shanthikumar (2007).

3. Conclusion

We have characterized the structure of optimal assignment rules as taking both allocative inefficiency and expenditure inefficiency into consideration. We have shown that the optimal structure depends crucially on the hazard rate of the value distribution F. Thus, if the hazard rate is decreasing (i.e., F is DFR), or equivalently $1 - F(x)$ is log-convex, then it is optimal to assign the object deterministically to the player with the highest valuation. On the other hand, if the hazard rate is increasing (i.e., F is IFR), or equivalently $1 - F(x)$ is log-concave, and all the expenditure counts as social costs (i.e., $\alpha = 1$) then it is optimal to assign the object randomly. Intermediate assignment rules may also be optimal that assign the object to one of the players in a strict subset, say a group of $n' < n$ players in decreasing order of valuation, with equal probability. We have also shown that both allocative performance and net efficiency of the optimal assignment rule increase as the variability of the value distribution increases in the sense that it changes from F to G with $G^{-1}F(x)/x$ increasing in x.

Acknowledgments

I thank Benny Moldovanu for helpful comments on an earlier version. I also thank anonymous referees for many invaluable suggestions. This work was supported by a National Research Foundation of Korea Grant funded by the Korean government (NRF-2010-330-B00085).

Appendix. A general mechanism design approach (with asymmetry)

We do not claim originality of the material in this appendix: It is a straightforward adaptation of Myerson's (1981) original auction design approach, though some care should be given to the payoff of the lowest possible type.[26] Hartline and Roughgarden (2008), Chakravarty and Kaplan (2009) and Condorelli (2011) have obtained similar derivations.

Let $N = \{1, 2, \ldots, n\}$ denote the set of players. Player i's valuation, denoted by v_i, is drawn from $V_i = [\underline{v}_i, \overline{v}_i]$ according to the distribution function F_i. We assume that $0 \leq \underline{v}_i \leq \overline{v}_i \leq \infty$, and that F_i admits a continuous and strictly positive density f_i. Assume also that v_i's for $i = 1, \ldots, n$ are independent. We use usual notation such as v, V, v_{-i}, and V_{-i} to denote the vector of valuations. In addition, let $\phi = f_1 \times f_2 \times \cdots \times f_n$, and use ϕ_{-i} as usual.

A direct mechanism is a pair $(\hat{p}, \hat{x})$ of outcome functions, with $\hat{p} : V \to \mathbb{R}^n_+$ and $\hat{x} : V \to \mathbb{R}^n_+$.[27] Hence, if the reported vector of valuations is v, then $\hat{p}_i(v)$ is the probability that i gets the object and $\hat{x}_i(v)$ is the level of expenditure that i exerts. By the revelation principle, it is with no loss of generality to restrict our attention to direct mechanisms. Players i's payoff is $\hat{p}_i v_i - x_i$ when he gets the object with probability $\hat{p}_i$, his valuation is v_i, and he exerts an expenditure of x_i. Hence, we assume that players are risk neutral and have additively separable utility functions.

Define

$$q_i(v_i) = \int_{V_{-i}} \hat{p}_i(v)\phi_{-i}(v_{-i})\,\mathrm{d}v_{-i}$$

to be the conditional expected probability assignment, and

$$y_i(v_i) = \int_{V_{-i}} \hat{x}_i(v)\phi_{-i}(v_{-i})\,\mathrm{d}v_{-i},$$

to be the conditional expected expenditure. If i believes that others will report truthfully, and i reports w_i when his true type is v_i, then his expected payoff is

$$U_i(w_i, v_i) = q_i(w_i)v_i - y_i(w_i).$$

The mechanism $(\hat{p}, \hat{x})$ is said to be *incentive compatible* if

$$U_i(v_i, v_i) \geq U_i(w_i, v_i), \quad \forall i \in N,\ \forall w_i, v_i \in V_i, \tag{A.1}$$

and *individually rational* if

$$U_i(v_i, v_i) \geq 0, \quad \forall i \in N,\ \forall v_i \in V_i. \tag{A.2}$$

In addition, we impose the probability condition

$$\hat{p}_i(v) \geq 0, \quad \forall i \in N \quad \text{and} \quad \sum_{j=1}^{n} \hat{p}_j(v) = 1, \quad \forall v \in V. \tag{A.3}$$

The mechanism $(\hat{p}, \hat{x})$ is said to be *feasible* if it satisfies (A.1)–(A.3). Following the well-established method originally in Myerson (1981), we have:

Lemma A. *The mechanism $(\hat{p}, \hat{x})$ is feasible if and only if the following conditions hold:*

$$w_i \leq v_i \Rightarrow q_i(w_i) \leq q_i(v_i), \quad \forall i \in N,\ \forall w_i, v_i \in V_i; \tag{A.4}$$

$$\begin{aligned} y_i(v_i) \;=\; & q_i(v_i)v_i - q_i(\underline{v}_i)\underline{v}_i + y_i(\underline{v}_i) \\ & - \int_{\underline{v}_i}^{v_i} q_i(w_i)\,\mathrm{d}w_i, \quad \forall i \in N,\ \forall v_i \in V_i; \end{aligned} \tag{A.5}$$

$$q_i(\underline{v}_i)\underline{v}_i - y_i(\underline{v}_i) \geq 0, \quad \forall i \in N; \tag{A.6}$$

together with

$$\hat{p}_i(v) \geq 0, \quad \forall i \in N \quad \textit{and} \quad \sum_{j=1}^{n} \hat{p}_j(v) = 1, \quad \forall v \in V. \tag{A.3}$$

Note well that this lemma implies that the monotonicity (A.4) must be satisfied in any incentive compatible assignment rule.

The objective in this paper is to maximize net efficiency

$$\begin{aligned}
&\int_V \left(\sum_{i=1}^{n} [\hat{p}_i(v)v_i - \alpha \hat{x}_i(v)] \right) \phi(v)\,\mathrm{d}v \\
&= \sum_{i=1}^{n} \int_{V_i} [q_i(v_i)v_i - \alpha y_i(v_i)] f_i(v_i)\,\mathrm{d}v_i \\
&= \sum_{i=1}^{n} \int_{V_i} \Bigg[q_i(v_i)v_i - \alpha \Bigg\{ q_i(v_i)v_i - q_i(\underline{v}_i)\underline{v}_i + y_i(\underline{v}_i) \\
&\quad - \int_{\underline{v}_i}^{v_i} q_i(w_i)\,\mathrm{d}w_i \Bigg\} \Bigg] f_i(v_i)\,\mathrm{d}v_i \\
&= \sum_{i=1}^{n} \int_{V_i} \Bigg[(1-\alpha)q_i(v_i)v_i + \alpha \int_{\underline{v}_i}^{v_i} q_i(w_i)\,\mathrm{d}w_i \\
&\quad - \alpha\{y_i(\underline{v}_i) - q_i(\underline{v}_i)\underline{v}_i\} \Bigg] f_i(v_i)\,\mathrm{d}v_i \\
&= \int_V \left(\sum_{i=1}^{n} \left[(1-\alpha)v_i + \alpha \frac{1 - F_i(v_i)}{f_i(v_i)} \right] \hat{p}_i(v) \right) \phi(v)\,\mathrm{d}v \\
&\quad - \alpha \sum_{i=1}^{n} \{y_i(\underline{v}_i) - q_i(\underline{v}_i)\underline{v}_i\}
\end{aligned}$$

---

[26] See (A.5) and (A.6) below and the discussion thereafter.

[27] We use the notation $(\hat{p}, \hat{x})$ to distinguish from similar notation in the text. The reason is that these functions are defined on the space of valuations, while the assignment rule in the text is defined on the space of expenditures.

subject to (A.3), (A.4) and (A.6). Now assume that $\underline{v}_i = 0$ for all $i = 1, \ldots, n$. Then, it is obvious that $y_i(\underline{v}_i) = 0$ for the maximization since $x_i \geq 0$. It is also obvious that (A.6) holds automatically. To summarize, the problem is to choose $(\hat{p}, \hat{x})$ to maximize

$$\int_V \left(\sum_{i=1}^n \left[(1-\alpha)v_i + \alpha \frac{1-F_i(v_i)}{f_i(v_i)} \right] \hat{p}_i(v) \right) \phi(v)\,\mathrm{d}v$$

subject to (A.3) and (A.4).

We now describe the optimal mechanism. Define the functions from [0, 1] to $\mathbb{R}$ as follows:

$$h_i(q) = (1-\alpha)F_i^{-1}(q) + \alpha \frac{1-q}{f_i(F_i^{-1}(q))},$$

$$H_i(q) = \int_0^q h_i(r)\,\mathrm{d}r,$$

$$G_i(q) = \mathrm{conv}H_i(q), \qquad g_i(q) = G_i'(q).$$

Note that G_i is the convex hull of the function H_i. As a convex function, G_i is continuously differentiable except at countably many points, and its derivative is monotone increasing. Define g_i as extending by right-continuity when G_i is not differentiable. Finally, let

$$\bar{c}_i(v_i) = g_i(F_i(v_i)) \quad \text{and}$$

$$M(v) = \left\{ i \in N \mid \bar{c}_i(v_i) = \max_{j\in N} \bar{c}_j(v_j) \right\}.$$

Observe that $G_i = H_i$, $g_i = h_i$, and $\bar{c}_i(v_i) = (1-\alpha)v_i + \alpha(1 - F_i(v_i))/f_i(v_i)$ when $(1-\alpha)v_i + \alpha(1-F_i(v_i))/f_i(v_i)$ is increasing. We have:

Proposition A. *Let $\bar{p} : V \to \mathbb{R}_+^n$ and $\bar{x} : V \to \mathbb{R}_+^n$ satisfy*

$$\bar{p}_i(v) = \begin{cases} 1/|M(v)| & \text{if } i \in M(v), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{x}_i(v) = \bar{p}_i(v)v_i - \int_{\underline{v}_i}^{v_i} \bar{p}_i(w_i, v_{-i})\,\mathrm{d}w_i.$$

Then, $(\bar{p}, \bar{x})$ represents an optimal mechanism.

This proposition gives us the following characterizations.

(1) When $\alpha = 0$: It is optimal to assign the object with probability one to the bidder with the highest v_i. That is, the winner-take-all assignment is optimal.
(2) When $\alpha = 1$: If the distribution F_i's are DFR, then it is optimal to assign the object with probability 1 to the bidder with the highest $(1 - F_i(v_i))/f_i(v_i)$. When players are symmetric so that $F_1 = F_2 = \cdots = F_n$, it is optimal to assign the object with probability 1 to the bidder with the highest v_i. On the other hand, if the distribution F_i's are IFR and players are symmetric, then it is optimal to assign the object with equal probability, that is, the random assignment is optimal.
(3) When $0 < \alpha < 1$: If the distribution F_i's are DFR and players are symmetric, then it is optimal to assign the object with probability 1 to the bidder with the highest v_i.

Hence, Propositions 2 and 3 in the text are still valid even when we consider more general mechanisms. Moreover, it is not hard to observe that the restriction to rank-order assignment rules may be binding only if $(1-\alpha)v_i + \alpha(1 - F_i(v_i))/f_i(v_i)$ is not monotonic with respect to v_i. In this case, the example below shows how the rank-order assignment rule may differ from the general optimal mechanism.

Example A. Let $\alpha = 1$ and $F_i(v_i) = \sqrt{v_i}$ on the support $[\underline{v}_i, \bar{v}_i] = [0, 1]$ for all $i = 1, \ldots, n$. Then, $(1 - F_i(v_i))/f_i(v_i) = 2\sqrt{v_i} - 2v_i$, $h_i(q) = 2q - 2q^2$ and $H_i(q) = q^2 - 2q^3/3$ on [0, 1]. Thus,

$$G_i(q) = \begin{cases} q^2 - \dfrac{2}{3}q^3 & \text{when } 0 \leq q \leq \dfrac{1}{4}, \\ -\dfrac{1}{24} + \dfrac{3}{8}q & \text{when } \dfrac{1}{4} \leq q \leq 1, \end{cases}$$

$$g_i(q) = \begin{cases} 2q - 2q^2 & \text{when } 0 \leq q \leq \dfrac{1}{4}, \\ \dfrac{3}{8} & \text{when } \dfrac{1}{4} \leq q \leq 1, \end{cases}$$

and

$$\bar{c}_i(v_i) = \begin{cases} 2\sqrt{v_i} - 2v_i & \text{when } 0 \leq v_i \leq \dfrac{1}{16}, \\ \dfrac{3}{8} & \text{when } \dfrac{1}{16} \leq v_i \leq 1. \end{cases}$$

When there are three players with $(v_1, v_2, v_3) = (1/2, 1/17, 1/18)$, it is optimal to assign the object to player 1, i.e., $\bar{p}_1(1/2, 1/17, 1/18) = 1$. On the other hand, when $(v_1, v_2, v_3) = (1/2, 1/15, 1/18)$, it is optimal to have $\bar{p}_1(1/2, 1/15, 1/18) = \bar{p}_2(1/2, 1/15, 1/18) = 1/2$. Therefore, the optimal assignment changes with players' valuations. Compare this with Example 2 in which the optimal rank-order assignment rule is $\pi_1 = \pi_2 = 1/2$, independent of players' valuations.

References

Amman, E., Leininger, W., 1995. Expected revenue of all-pay and first-price sealed-bid auctions with affiliated signals. Journal of Economics 61, 273–279.
Amman, E., Leininger, W., 1996. Asymmetric all-pay auctions with incomplete information: the two-payer case. Games and Economic Behavior 14, 1–18.
Bagnoli, M., Bergstrom, T., 2005. Log-concave probability and its applications. Journal of Economic Theory 26, 445–469.
Barlow, R., Proschan, F., 1966. Inequalities for linear combinations of order statistics from restricted families. Annals of Mathematical Statistics 37, 1574–1592.
Bogomolnaia, A., Moulin, H., 2001. A new solution to the random assignment problem. Journal of Economic Theory 100 295–328.
Budish, E., Che, Y., Kojima, F., Milgrom, P., 2011. Designing random allocation mechanisms: theory and applications. Manuscript.
Chakravarty, S., Kaplan, T., 2009. Optimal allocation without transfer payments. Manuscript.
Condorelli, D., 2011. What money can't buy: optimal allocations with priority lists, lotteries and queues. Manuscript.
David, H., 1981. Order Statistics. John Wiley and Sons Inc, New York.
Frankel, A., 2010. Aligned delegation. Manuscript.
Green, J., Stokey, N., 1983. A comparison of tournaments and contracts. Journal of Political Economy 91, 349–364.
Hartline, J., Roughgarden, T., 2008. Optimal mechanism design and money burning. STOC'08.
Hoppe, H.C., Moldovanu, B., Sela, A., 2009. The theory of assortative matching based on costly signals. Review of Economic Studies 76, 253–281.
Hylland, A., Zeckhauser, R., 1979. The efficient allocation of individuals to positions. Journal of Political Economy 87, 293–314.
Koh, W., Yang, Z., Zhu, L., 2006. Lottery rather than waiting-line auction. Social Choice and Welfare 27, 289–310.
Lazear, E., Rosen, S., 1981. Rank-order tournaments as optimal labor contracts. Journal of Political Economy 89, 841–864.
Moldovanu, B., Sela, A., Shi, X., 2007. Contests for status. Journal of Political Economy 115, 338–363.
Moldovanu, B., Sela, A., Shi, X., 2008. Comparing auctions with endogenous quantities. Journal of Economic Theory 141, 1–27.
Myerson, R., 1981. Optimal auction design. Mathematics of Operations Research 6, 58–73.
Patel, J., Kapadia, C., Owen, D., 1976. Handbook of Statistical Distributions. Marcel Dekker, Inc, New York.
Shaked, M., Shanthikumar, J.G., 2007. Stochastic Orders. Springer, New York.
Spence, M., 1973. Job market signaling. Quarterly Journal of Economics 87, 296–332.
Suen, W., 1989. Rationing and rent dissipation in the presence of heterogeneous individuals. Journal of Political Economy 97, 1384–1394.
Taylor, G., Tsui, K., Zhu, L., 2003. Lottery or waiting-line auction? Journal of Public Economics 87, 1313–1334.
Tullock, G., 1980. Efficient rent-seeking. In: Buchanan, J., Tollison, R., Tullock, G. (Eds.), Towards a Theory of the Rent-Seeking Society. Texas A&M University Press, College Station, pp. 97–112.