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The effective minimax value of asynchronously repeated games*

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Abstract. We study the effect of asynchronous choice structure on the possibility of cooperation in repeated strategic situations. We model the strategic situations as asynchronously repeated games, and define two notions of effective minimax value. We show that the order of players' moves generally affects the effective minimax value of the asynchronously repeated game in significant ways, but the order of moves becomes irrelevant when the stage game satisfies the non-equivalent utilities (NEU) condition. We then prove the Folk Theorem that a payoff vector can be supported as a subgame perfect equilibrium outcome with correlation device if and only if it dominates the effective minimax value. These results, in particular, imply both Lagunoff and Matsui's (1997) result and Yoon (2001)'s result on asynchronously repeated games.

Key words: Effective minimax value, folk theorem, asynchronously repeated games

1. Introduction

Asynchronous choice structure in repeated strategic situations may affect the possibility of cooperation in significant ways. When players in a repeated game make asynchronous choices, that is, when players cannot always change their actions simultaneously in each period, it is quite plausible that they can coordinate on some particular actions via short-run commitment or inertia to render unfavorable outcomes infeasible. Indeed, Lagunoff and Matsui (1997) showed that, when a pure coordination game is repeated in a way that only

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one player can move in each period, the only subgame perfect equilibrium payoff with correlation device (when players are sufficiently patient) is the one that Pareto dominates all other payoffs.

On the other hand, a recent paper by Yoon (2001) for asynchronously repeated games proves the Folk Theorem that every feasible and strictly individually rational payoff vector can be supported as a subgame perfect equilibrium outcome with correlation device as long as the non-equivalent utilities (NEU) condition together with the finite periods of inaction (FPI) condition (or more generally, the finite periods of inaction in expectation (FPIE) condition for stochastic environments) is satisfied. Since the FPI condition (or the FPIE condition), which requires that the length of periods during which every player has at least one opportunity to change his action is finite (or finite in expectation), only imposes a normality condition for a repeated strategic situation,[1] the divergence of Yoon's result from Lagunoff and Matsui's hinges solely on the NEU condition. Recall that the NEU condition precludes the existence of common interests between any two players: The NEU condition requires that no pair of players have equivalent payoffs.

In this paper, we pursue this line of research further to unify the existing results. To state the main result informally, we introduce two notions of effective minimax value for repeated games which explicitly incorporate the order of moves, and (essentially) prove the Folk Theorem that a payoff vector can be supported as a subgame perfect equilibrium outcome with correlation device if and only if it dominates the effective minimax value of the repeated game.[2] This result, therefore, dispenses with the NEU condition required for the Folk Theorem in Yoon (2001). Recall that the effective minimax value, which was first introduced in Wen (1994), for a stage game is a generalization of the standard minimax value in that it takes equivalent utilities among players into account.[3] We, in turn, generalize the effective minimax value to asynchronously repeated games (with a stationarity condition imposed).[4]

We first determine the reservation value, i.e., the minimum level of payoffs that a player can guarantee to himself in an asynchronously repeated game. We introduce two notions of effective minimax value, the *upper effective minimax value* and the *lower effective minimax value*. We state in Theorem 1 the properties of these effective minimax values in relation to the minimax value of the stage game. We first find that the order of moves affects the effective minimax values of the repeated game in significant ways. However, when the stage game satisfies the NEU condition, it becomes irrelevant in the sense that the same upper effective minimax value obtains regardless of the

[1] If a repeated game does not satisfy the FPI condition or the FPIE condition, then it is not a repeated game in a true sense. See Yoon (2001) for precise definitions.

[2] As will become clear in the next section, this paper deals with the infinitely repeated games with discounting as opposed to no discounting, and the solution concept employed is subgame perfect equilibrium with correlation device as opposed to Nash equilibrium.

[3] See the next section for a precise definition of these concepts.

[4] In a related paper, Wen (2002a) generalizes the effective minimax value to repeated sequential games, where sequential games are a special class of extensive form games in which disjoint subsets of players move sequentially. Payoffs are realized at the end of one turn of players' moves in sequential games. In contrast, payoffs are realized at the end of each period in asynchronously repeated games, which makes the job of defining a proper effective minimax value challenging.

order of moves. Moreover, the upper effective minimax value of the repeated game coincides with the effective minimax value of the stage game (which in turn coincides with the standard minimax value). We also find that, if the repeated game is a simultaneous-move game so that every player can move in each period, then both the upper and the lower minimax values of the repeated game coincide with the effective minimax value of the stage game.

We next show that the effective minimax values are meaningful bounds for the Folk Theorem. We prove in Theorem 2 that any subgame perfect equilibrium payoff with correlation device in an asynchronously repeated game dominates the lower effective minimax value. This result, in particular, implies the result of Lagunoff and Matsui (1997) since it can be easily shown that the lower effective minimax value of the repeated game when the stage game is a pure coordination game is players' maximum possible payoff. In Theorem 3, we prove the Folk Theorem that any payoff vector can be supported as a subgame perfect equilibrium outcome with correlation device as long as it dominates the upper effective minimax value of the repeated game.

The paper is organized as follows. We describe asynchronously repeated games in the next section. We define two notions of effective minimax value of the repeated game, and then state and prove main results in Section 3. Finally, we discuss some limitations of the present paper and directions for future research in Section 4.

2. The model

2.1. The stage game

Let $G = \left(I, (A_i)_{i=1}^n, (u_i)_{i=1}^n\right)$ denote a strategic form game where $I = \{1, \ldots, n\}$ is the set of players, A_i is the set of *actions* for player i, and u_i is the stage game payoff function from $A = \times_{i=1}^n A_i$ to $\mathbb{R}$. The payoff vector $u = (u_1, \ldots, u_n)$ is a function from A to $\mathbb{R}^n$. We will assume that A_i's are compact sets and that the u_i's are continuous. The set A_i may be interpreted as the (either finite or infinite) set of *pure* actions. We note that many game-theoretic models of application involve continuous action variables and that players in these models use only pure actions, not probability distributions over the continuous variables. See, for example, Maskin and Tirole (1988), where an asynchronously repeated game of oligopoly is studied. Alternatively, A_i may be interpreted as the set of *mixed* actions, i.e., probability distributions, defined over the underlying finite set of pure actions.[5] In the latter interpretation, it is assumed that mixed actions are observable and implementable as they are.[6]

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[5] That is, $A_i = \Delta(B_i)$ where B_i is a finite set of actions.

[6] Fudenberg and Maskin (1986) and Abreu, Dutta, and Smith (1994) for infinitely repeated games with discounting show that it is with no loss of generality to treat the unobservable mixed actions virtually as the observable pure actions. On the other hand, Benoit and Krishna (1985) for finitely repeated games prove the Folk Theorem only for the observable pure actions case, which is later extended in Gossner (1995) to the unobservable mixed actions case. In this paper, we follow Benoit and Krishna (1985) in preferring the pure actions interpretation, while leaving the unobservable mixed actions case for future research. Note that, under the mixed actions interpretation, one needs to construct repeated game strategies and also attain the target payoff vector based only on the histories of pure actions (i.e., on the realizations of mixed actions), without the assumption of ex post observability of mixed actions.

Following the convention, let V be the convex hull of the set of feasible payoff vectors $\{u(a) : a \in A\}$.

Let a_{-i} denote an action profile of player i's opponents (that is, players in $I \setminus \{i\}$). More generally, for a given subset J of players, let a_J denote an action profile of players in J. Player i's *standard minimax value* is defined as

$$\underline{v}_i = \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i}).$$

We call $(\underline{v}_1, \ldots, \underline{v}_n)$ the standard minimax point of G. A payoff vector $v = (v_1, \ldots, v_n)$ is called feasible and strictly individually rational if $v \in V$ and $v_i > \underline{v}_i$ for all $i \in I$.

Following Abreu, Dutta and Smith (1994), we now define an equivalence relation on I as follows: We write $i \sim j$ and say player i and player j have equivalent utilities if there exist c_{ij} and $d_{ij} > 0$ such that $u_i(a) = c_{ij} + d_{ij}u_j(a)$ for all $a \in A$. The equivalence relation $\sim$ induces a partition, say E, of the set of players, and we use $E(i)$ to denote a member of E such that $E(i) = \{j \in I | i \sim j\}$. That is, $E(i)$ is the set of players who have equivalent utilities. Player i's *effective minimax value*, which was first introduced in Wen (1994), is defined as

$$\underline{v}_i^e = \min_{a} \max_{j \in E(i)} \max_{a_j'} u_i(a_j', a_{-j}).$$

We call $(\underline{v}_1^e, \ldots, \underline{v}_n^e)$ the effective minimax point of G.

The effective minimax value is not less than the standard minimax value, i.e., $\underline{v}_i^e \geq \underline{v}_i$. In addition, when the nonequivalent utilities (NEU) condition of Abreu, Dutta, and Smith (1994) holds, that is, $E(i) = \{i\}$ for all $i \in I$, the effective minimax value is equal to the standard minimax value.

2.2. Asynchronously repeated games

We now describe asynchronously repeated games. The stage game is repeated in periods $t = 0, 1, \ldots$. In period 0, all players choose their respective actions $a = (a_1, \ldots, a_n)$. In period $t \geq 1$, a subset $I_t \subseteq I$ of players (I_t may be $\emptyset$) are able to change their actions $a_{I_t} = a|_{I_t}$. In asynchronously repeated games, I_t is a realization of the 2^I-valued random variable $\tilde{I}_t$ whose distribution is a function of the sequence $(I_1, \ldots, I_{t-1})$ of past realizations of $\tilde{I}_\tau$'s, $\tau = 1, \ldots, t-1$, and the sequence $(a^0, \ldots, a^{t-1})$ of past action profiles. We impose in this paper a stationarity condition on asynchronously repeated games: There exists $\bar{t}$ such that

(i) $\bigcup_{t=1}^{\bar{t}} I_t = I$, and
(ii) $I_{m\bar{t}+s} = I_s$ for all $m = 1, 2, \ldots$ and $s = 1, \ldots, \bar{t}$.

In addition to (i) and (ii), we require that (iii) there does not exist $t' < \bar{t}$ which satisfies (i) and (ii). That is, $\bar{t}$ is the minimum number of periods which satisfies (i) and (ii). In other words, we assume that the same sequence of moves is repeated in every $\bar{t}$ periods.[7] We note that the sequence $(I_1, \ldots, I_{\bar{t}})$

[7] This stationarity condition implies the finite periods of inaction (FPI) condition of Yoon (2001). We admit that the current condition is somewhat restrictive. We remark, however, that some form of stationarity condition should be imposed on the asynchronously repeated game if we want to obtain meaningful relationships among various minimax values.

need not partition I. As a concrete example, the game in which $I = \{1, 2, 3\}$, $\bar{t} = 3$, $I_{3m+1} = \{1, 2\}$, $I_{3m+2} = \{2, 3\}$, and $I_{3m+3} = \{1, 3\}$ is an asynchronously repeated game which satisfies the stationarity condition. This framework includes many interesting repeated strategic situations.

Example 1. (Simultaneous-move games) $\bar{t} = 1$ and $I_1 = I$. This is the environment studied in traditional repeated games, as in Fudenberg and Maskin (1986) or Abreu, Dutta, and Smith (1994).

Example 2. (Round-robin-move games) $\bar{t} = n$ and $I_t = \{t\}$ for $t = 1, \ldots, n$. These games canonically represent alternating-move games studied in Lagunoff and Matsui (1997).

In what follows, we will call an asynchronously repeated game that satisfies the stationarity condition simply as a repeated game and denote it by a tuple $\Gamma = (G, \{I_t\}_{t=1}^{\bar{t}}, \delta)$, where δ is the common discount factor.

2.3. Strategies

A history at the beginning of period t, a t-history, is $h(t) = (a^0, a^1, \ldots, a^{t-1})$. Let $H(t)$ be the set of all t-histories. A strategy for player i is a sequence of functions $\sigma_i = \{\sigma_i^t\}_{t=0}^{\infty}$ such that

(1) $\sigma_i^0 \in A_i$, and
(2) For $t \geq 1$,

$$\begin{cases} \sigma_i^t : H(t) \to A_i & \text{if } i \in I_t, \\ \sigma_i^t = a_i^{t-1} & \text{otherwise.} \end{cases}$$

Each strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ generates a distribution over the sequences of the stage-game payoff vectors. Thus if $\{g_i^t\}$ is player i's sequence of stage-game payoffs, his objective in the repeated game is to maximize the expected value of the discounted *average* payoff

$$(1 - \delta) \sum_{t=0}^{\infty} \delta^t g_i^t.$$

The equilibrium concept we employ is subgame perfect equilibrium with correlation device.

3. The effective minimax value of asynchronously repeated games

Suppose the NEU condition is satisfied. The Folk Theorem of Yoon (2001) proves that any feasible and strictly individually rational payoff vector can be supported as a subgame perfect equilibrium outcome with correlation device of an asynchronously repeated game as long as the FPI condition (or the FPIE condition) is met, *regardless of the order of moves*. We will show in the present section that, if the NEU condition is not satisfied, then the exact order of moves becomes significant in determining the set of subgame perfect equilibrium payoff vectors with correlation device.

Consider a repeated game $\Gamma = (G, \{I_t\}_{t=1}^{\bar{t}}, \delta)$. We define a *move pattern* for Γ as a sequence of consecutive subsets of players, of length $\bar{t}$. By the stationarity condition, there are exactly $\bar{t}$ possible move patterns in Γ, namely, $(I_1, \ldots, I_{\bar{t}})$, $(I_2, \ldots, I_{\bar{t}}, I_1)$, $\ldots, (I_t, I_{t+1}, \ldots, I_{\bar{t}}, I_1, \ldots, I_{t-1}), \ldots$, and $(I_{\bar{t}}, I_1, \ldots, I_{\bar{t}-1})$. Let $\mathscr{I}$ be the collection of all move patterns and fix a move pattern $(I(1), \ldots, I(\bar{t})) \in \mathscr{I}$. We define, for each $k \in \{1, \ldots, \bar{t}\}$,[8]

$$I_k^b = \bigcup_{k'=1}^{k-1} I(k'), \quad I_k^a = \bigcup_{k'=k+1}^{\bar{t}} I(k').$$

I_k^b (I_k^a, respectively) is the set of players who move before (after, respectively) step k in a given move pattern $(I(1), \ldots, I(\bar{t})) \in \mathscr{I}$. Player i's effective minimax value in Γ given a move pattern $(I(1), \ldots, I(\bar{t}))$ is defined recursively as follows.

In the following description, we use the notation $u_i(a_J; a_{J'})$ for subsets J and J' of players, with $J \cup J' = I$. This is player i's payoff when players in J' choose the action profile $a_{J'}$ and players in J choose the action profile a_J. When J and J' are not disjoint, the convention is that the corresponding components of a_J *override* those of $a_{J'}$. As a concrete example, consider the case when $I = \{1, 2\}$, $J = \{1\}$, and $J' = \{1, 2\}$. Let $a_J = a_1$ and $a_{J'} = (\hat{a}_1, \hat{a}_2)$. Then, $a_J = a_1$ overrides the first element of $a_{J'} = (\hat{a}_1, \hat{a}_2)$. We thus have $u_i(a_J; a_{J'}) = u_i(a_1, \hat{a}_2)$.

First, in step $\bar{t}$ and given $a_{I_{\bar{t}}^b}$, consider the following minimax problem

$$\min_{a_{I(\bar{t})}} \max_{j \in E(i) \cap I(\bar{t})} \max_{a'_j} u_i(a_{I(\bar{t})}; a_{I_{\bar{t}}^b}).$$

This problem finds an action profile $a_{I(\bar{t})}$ for players in $I(\bar{t})$ given the action profile $a_{I_{\bar{t}}^b}$ for players in $I_{\bar{t}}^b$. It is possible that the two sets $I(\bar{t})$ and $I_{\bar{t}}^b$ are not disjoint. In this case, the corresponding components of action profile $a_{I(\bar{t})}$ *override* those of $a_{I_{\bar{t}}^b}$, so that we will always get an action profile of dimension $|I|$ as an argument of the payoff function $u_i(\cdot)$. The way we find a solution profile for $I(\bar{t})$ is essentially the same as that of finding an action profile that achieves the effective minimax value of the stage game: Find an $a_{I(\bar{t})}$ that minimizes $u_i(\cdot)$ while allowing players in $E(i) \cap I(\bar{t})$ to choose an action that maximizes $u_i(\cdot)$. The only difference is that we change actions of players in $I(\bar{t})$ while keeping others' actions fixed. This is so because only players in $I(\bar{t})$ can move in step $\bar{t}$. By altering $a_{I_{\bar{t}}^b}$, we can get a function $\beta_{i,\bar{t}} : \times_{j \in I_{\bar{t}}^b} A_j \to \times_{j \in I(\bar{t})} A_j$, which assigns an action profile for $I(\bar{t})$ for each action profile of $I_{\bar{t}}^b$.

Next, in step $(\bar{t}-1)$ and given $a_{I_{\bar{t}-1}^b}$ and $\beta_{i,\bar{t}}(\cdot)$, consider

$$\min_{a_{I(\bar{t}-1)}} \max_{j \in E(i) \cap I(\bar{t}-1)} \max_{a'_j} u_i(\beta_{i,\bar{t}}(a_{I(\bar{t}-1)}; a_{I_{\bar{t}-1}^b}); a_{I(\bar{t}-1)}; a_{I_{\bar{t}-1}^b}).$$

This problem finds an action profile $a_{I(\bar{t}-1)}$ for players in $I(\bar{t}-1)$ given the action profile $a_{I_{\bar{t}-1}^b}$ for players in $I_{\bar{t}-1}^b$ and the function $\beta_{i,\bar{t}}(\cdot)$ found in the previous step. Here again, if two sets $I(\bar{t}-1)$ and $I_{\bar{t}-1}^b$ are not disjoint then the corresponding components of action profile $a_{I(\bar{t}-1)}$ override those of $a_{I_{\bar{t}-1}^b}$. Note that the resulting action profile, denoted by $(a_{I(\bar{t}-1)}; a_{I_{\bar{t}-1}^b})$ and which

[8] The backward sequential formulation of minimax problems below is adopted from Wen (2002a)'s formulation for sequential games.

becomes an action profile $a_{I^b_{\bar{t}}}$ for step $\bar{t}$, may not be of dimension $|I|$. We now combine this profile with $\beta_{i,\bar{t}}(a_{I(\bar{t}-1)}; a_{I^b_{\bar{t}-1}})$ to produce an action profile of dimension $|I|$, where components of $\beta_{i,\bar{t}}(a_{I(\bar{t}-1)}; a_{I^b_{\bar{t}-1}})$ override those of $(a_{I(\bar{t}-1)}; a_{I^b_{\bar{t}-1}})$ if necessary. By altering $a_{I^b_{\bar{t}-1}}$, we can get a function $\beta_{i,\bar{t}-1} : \times_{j\in I^b_{\bar{t}-1}} A_j \to \times_{j\in I(\bar{t}-1)\cup I(\bar{t})} A_j$, which assigns an action profile for $I(\bar{t}-1)\cup I(\bar{t})$ for each action profile of $I^b_{\bar{t}-1}$.

Generally, in step k ($1 \le k \le \bar{t}-1$) and given $a_{I^b_k}$ and $\beta_{i,k+1}(\cdot)$, consider the following minimax problem

$$\min_{a_{I(k)}} \max_{j\in E(i)\cap I(k)} \max_{a'_j} u_i(\beta_{i,k+1}(a_{I(k)}; a_{I^b_k}); a_{I(k)}; a_{I^b_k}).$$

This problem finds an action profile $a_{I(k)}$ for players in $I(k)$ given the action profile $a_{I^b_k}$ for players in I^b_k and the function $\beta_{i,k+1}(\cdot)$ found in the previous step. We proceed as before. By altering $a_{I^b_k}$, we can get a function $\beta_{i,k} : \times_{j\in I^b_k} A_j \to \times_{j\in I(k)\cup I^a_k} A_j$, which assigns an action profile for $I(k)\cup I^a_k$ for each action profile of I^b_k.

We observe that the value of minimax problem in step k is not higher than that in step $k+1$. Formally, we have:

Lemma 1. *For any given move pattern $(I(1),\dots,I(\bar{t})) \in \mathscr{I}$ and for any $a_{I^b_k}$ and $\beta_{i,k+1}(\cdot)(k=1,\dots,\bar{t}-1)$, we have*

$$u_i(\beta_{i,k}(a_{I^b_k}); a_{I^b_k}) \le u_i(\beta_{i,k+1}(a_{I^b_{k+1}}); a_{I^b_{k+1}})$$

where $a_{I^b_{k+1}} = (\tilde{a}_{I(k)}; a_{I^b_k})$ and $\tilde{a}_{I(k)}$ is a solution to step k minimax problem given $a_{I^b_k}$ and $\beta_{i,k+1}(\cdot)$.

Proof: We have

$$\begin{aligned} u_i(\beta_{i,k}(a_{I^b_k}); a_{I^b_k}) &= \min_{a_{I(k)}} \max_{j\in E(i)\cap I(k)} \max_{a'_j} u_i(\beta_{i,k+1}(a_{I(k)}; a_{I^b_k}); a_{I(k)}; a_{I^b_k}) \\ &\le \max_{j\in E(i)\cap I(k)} \max_{\overline{a}'_j} u_i(\beta_{i,k+1}(\overline{a}_{I(k)}; a_{I^b_k}); \overline{a}_{I(k)}; a_{I^b_k}) \end{aligned}$$

for all $a_{I^b_k}$ and $\overline{a}_{I(k)}$. Let $\hat{a}_{I(k)}$ be a solution to the last maximax problem. (That is, $\hat{a}_{I(k)}$ is obtained from $\overline{a}_{I(k)}$ by replacing $\overline{a}_j$ with a maximizer $\hat{a}_j$.) Then,

$$u_i(\beta_{i,k}(a_{I^b_k}); a_{I^b_k}) \le u_i(\beta_{i,k+1}(\hat{a}_{I(k)}; a_{I^b_k}); \hat{a}_{I(k)}; a_{I^b_k}).$$

Since this inequality holds for all $a_{I^b_k}$ and $\overline{a}_{I(k)}$, we are done. ■

Player i's effective minimax value of the repeated game Γ given a move pattern $(I(1),\dots,I(\bar{t}))$, which is denoted by $\underline{r}^e_i(I(1),\dots,I(\bar{t}))$, is the value of the previous minimax problem applied to step 1. We introduce two notions of effective minimax value of Γ.

Definition 1. (Upper effective minimax value) Player i's *upper effective minimax value* of the repeated game Γ is defined as

$$\underline{r}^e_i = \max_{\mathscr{I}} \underline{r}^e_i(I(1),\dots,I(\bar{t})),$$

and we call $(\underline{r}^e_1,\dots,\underline{r}^e_n)$ the upper effective minimax point of Γ.

Definition 2. (Lower effective minimax value) Player i's *lower effective minimax value* of the repeated game Γ is defined as

$$\underline{s}_i^e = \min_{\mathscr{I}} \underline{r}_i^e(I(1), \ldots, I(\bar{t})),$$

and we call $(\underline{s}_1^e, \ldots, \underline{s}_n^e)$ the lower effective minimax point of Γ.

Since there are only a finite number of move patterns in Γ, these two values are well-defined and obviously $\underline{s}_i^e \leq \underline{r}_i^e$. We state in Theorem 1 the relationship among different minimax values defined so far, i.e., $\underline{v}_i$, $\underline{v}_i^e$, $\underline{r}_i^e$, and $\underline{s}_i^e$.

Theorem 1. (*i*) *We have* $\underline{r}_i^e \geq \underline{v}_i^e \geq \underline{v}_i$ *for any repeated game* $\Gamma = (G, \{I_t\}_{t=1}^{\bar{t}}, \delta)$.
(*ii*) *If the stage game G satisfies the NEU condition, then* $\underline{r}_i^e$ *remains constant across all repeated games* $\Gamma = (G, \{I_t\}_{t=1}^{\bar{t}}, \delta)$ *with fixed G. Moreover,* $\underline{r}_i^e = \underline{v}_i^e = \underline{v}_i = \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i})$.
(*iii*) *If the repeated game is a simultaneous-move game, i.e.,* $\bar{t} = 1$, *then* $\underline{r}_i^e = \underline{s}_i^e = \underline{v}_i^e \geq \underline{v}_i$.

Proof: (i) It follows almost immediately from the definition.
(ii) Consider a move pattern $(I(1), \ldots, I(\bar{t}))$ such that $i \in I(\bar{t})$. Since the stage game satisfies the NEU condition, we have $\underline{r}_i^e = \underline{r}_i^e(I(1), \ldots, I(\bar{t}))$ and, moreover, this is equal to $\underline{v}_i^e = \underline{v}_i = \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i})$ for any repeated game Γ.
(iii) Obvious. ■

Theorem 1(ii) states that, as long as the NEU condition is satisfied, the exact order of moves is irrelevant in the sense that the same upper effective minimax value obtains regardless of the order of moves. (Note that we do not use δ in defining the minimax values. Hence, these values are independent of δ.) In particular, the upper effective minimax value of the repeated game is equal to the minimax value of the stage game. It is also not hard to see that, if the NEU condition is satisfied and the repeated game is a single-move game,[9] then the lower effective minimax value of the repeated game is in fact the maximin value $\max_{a_i} \min_{a_{-i}} u_i(a_i, a_{-i})$ of the stage game. In addition, Theorem 1(ii)–(iii) together imply that $\underline{r}_i^e = \underline{v}_i^e = \underline{s}_i^e = \underline{v}_i$ when the NEU condition is satisfied and the repeated game is a simultaneous-move game. On the other hand, when the NEU condition is not satisfied and the repeated game is not a simultaneous-move game, we may have $\underline{r}_i^e > \underline{v}_i^e$ and even $\underline{s}_i^e > \underline{v}_i^e$.

Example 3. (Pure coordination game) Consider the following pure coordination game, where player 1 chooses rows and player 2 chooses columns.[10]

	L	R
U	3,3	1,1
D	2,2	0,0

Then, $\underline{v}_1 = 1$ and an action profile that achieves $\underline{v}_1$ is (U,R), while $\underline{v}_1^e = 2$ and an action profile that achieves $\underline{v}_1^e$ is (D,L). When players take alternating

[9] A single-move game is an asynchronously repeated game in which I_t is a singleton for all t.

[10] Note that all the players have equivalent utilities in pure coordination games.

moves, that is, when the game is a repeated game in which $\bar{t} = 2$ and $I_t = \{t\}$ for $t = 1, 2$, we have $\underline{r}_1^e = \underline{s}_1^e = 3$.

More generally, we have:

Proposition 1. *Consider a pure coordination game played by n players where $u_i(a) = u_j(a)$ for all $i, j \in I$ and $a \in A$, and let $u^* = \max_{a \in A} u_i(a)$ for all $i \in I$ be the highest payoff each player can get in this game. If the repeated game is a single-move game, i.e., if only one player can move in each period, then* $\underline{r}_i^e = \underline{s}_i^e = u^*$ for all $i \in I$.

Proof: Obvious. ■

We next provide an example which shows that, when the NEU condition is not satisfied, then the order of moves may become important in determining the effective minimax values.

Example 4. (3 player game) Consider the following 3 player game, where player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. In this example, we assume that mixed actions are observable and implementable.

	L	R
U	3, 3, 0	1, 1, 0
D	2, 2, 0	0, 0, 0

A

	L	R
U	0, 0, 0	2, 2, 0
D	1, 1, 0	3, 3, 0

B

Then, $\underline{v}_1 = 1$ and an action profile that achieves $\underline{v}_1$ is (U,R,A), while $\underline{v}_1^e = 1.5$ and an action profile that achieves $\underline{v}_1^e$ is $(\frac{1}{2}\text{U} + \frac{1}{2}\text{D}, \frac{1}{2}\text{L} + \frac{1}{2}\text{R}, \frac{1}{2}\text{A} + \frac{1}{2}\text{B})$. When the repeated game is such that $\bar{t} = 2$, $I_1 = \{1, 2\}$, and $I_2 = \{3\}$, we have $\underline{r}_1^e = \underline{r}_1^e(\{3\}, \{1, 2\}) = 1.5$ and $\underline{s}_1^e = \underline{r}_1^e(\{1, 2\}, \{3\}) = 1$. On the other hand, when the repeated game is such that $\bar{t} = 2$, $I_1 = \{1, 3\}$, and $I_2 = \{2\}$, we have $\underline{r}_1^e = \underline{s}_1^e = 1.5$.

Note that players 1 and 2 have equivalent utilities in this example. Therefore, when the repeated game is such that $I_1 = \{1, 3\}$, and $I_2 = \{2\}$, players 1 and 2 can coordinate over time to attain a value of 1.5 whatever the move pattern is. Moreover, this value is clearly equal to the effective minimax value of the stage game. On the other hand, when the repeated game is such that $I_1 = \{1, 2\}$, and $I_2 = \{3\}$, they can only attain their joint maximin value of 1 under the move pattern $(\{1, 2\}, \{3\})$ since they cannot coordinate over time.[11]

What is the reservation value for player i in a repeated game $\Gamma = (G, \{I_t\}_{t=1}^{\bar{t}}, \delta)$, that is, the minimum level of payoffs that player i can guarantee to himself in Γ when he correctly anticipates others' action choices?

[11] As the example reveals, what really matters in determining the effective minimax values is the order of final moves. In general, the effective minimax value given a move pattern depends only on the following parameter. Let t_i be the last time in the move pattern that player i may change his action, namely, $t_i = \max\{t | i \in I(t)\}$. Then the effective minimax value given one move pattern is equal to the effective minimax value given another move pattern as long as the *order* of t_i's are the same across two move patterns. This is so since the actions chosen in later periods can always override the ones before. I thank a referee for pointing out this observation.

The natural candidate is of course the upper minimax value $\underline{r}_i^e$ since it corresponds to the minimax value of the stage game when the NEU condition is satisfied. (See Theorem 1(ii) above.) We show, however, that the non-simultaneity of the moves in a repeated game may hold a player's payoff below $\underline{r}_i^e$.

Example 5. (Reservation value) Consider the following two player game, where player 1 chooses rows and player 2 chooses columns. In this example, we will restrict the attention to pure actions for ease of presentation.

	L	R
U	a, 0	0, 0
D	0, 0	a, 0

When players take alternating moves and $a > 0$, we have $\underline{r}_1^e = \underline{r}_1^e(\{2\},\{1\}) = \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i}) = a$ and $\underline{s}_1^e = \underline{r}_1^e(\{1\},\{2\}) = \max_{a_i} \min_{a_{-i}} u_i(a_i, a_{-i}) = 0$. Suppose now that player 2 chooses R when player 1 chooses U in the previous period, and player 2 chooses L when player 1 chooses D in the previous period. Even if player 1 correctly anticipates his opponent's action choices, his maximum payoff against player 2's strategy is $a/2$ since he cannot move in periods player 2 moves. In Theorem 2 below, we show that a lower bound for the right reservation value in a repeated game is the lower effective minimax value $\underline{s}_i^e$.

Theorem 2. *For any* $\epsilon > 0$, *there exists a discount factor* $\underline{\delta} < 1$ *such that, for all* $\delta \in (\underline{\delta}, 1)$, *player i's subgame perfect equilibrium payoffs with correlation device in the repeated game* $\Gamma = (G, \{I_t\}_{t=1}^{\bar{t}}, \delta)$ *are not less than* $\underline{s}_i^e - \epsilon$.

Proof: Define $w_i = \min_a u_i(a)$, and let L_i be the infimum of player i's subgame perfect equilibrium continuation payoffs with correlation device in Γ. Since L_i is the infimum, there exist a period t and player i's equilibrium continuation payoff in t denoted by V_t^i such that $L_i > V_i^t - \eta$ for any $\eta > 0$. Now, consider a move pattern which ends in period t, that is, $(I(1), \ldots, I(\bar{t}))$ with $I(\bar{t}) = I_t$.[12] We first observe that, for any $a_{I_{\bar{t}}^b}$ in step $\bar{t}$ of the given move pattern and for any $\delta \in (0,1)$,

$$V_i^t \geq (1-\delta) \min_{a_{I(\bar{t})}} \max_{j \in E(i) \cap I(\bar{t})} \max_{a_j'} u_i(a_{I(\bar{t})}; a_{I_{\bar{t}}^b}) + \delta L_i,$$

by the equivalent utilities relation. Hence, we get

$$L_i > \min_{a_{I(\bar{t})}} \max_{j \in E(i) \cap I(\bar{t})} \max_{a_j'} u_i(a_{I(\bar{t})}; a_{I_{\bar{t}}^b}) - \frac{\eta}{1-\delta}.$$

Since this inequality holds for any $\eta > 0$, we obtain

$$L_i \geq \min_{a_{I(\bar{t})}} \max_{j \in E(i) \cap I(\bar{t})} \max_{a_j'} u_i(a_{I(\bar{t})}; a_{I_{\bar{t}}^b}) = u_i(\beta_{i,\bar{t}}(a_{I_{\bar{t}}^b}); a_{I_{\bar{t}}^b}).$$

Next, for any $a_{I_{\bar{t}-1}^b}$ in step $(\bar{t}-1)$ of the given move pattern and for any $\delta \in (0,1)$, player i's continuation payoffs are not less than

[12] We assume that $t \geq \bar{t}$ in the proof. Observe that it is easier to prove when $t < \bar{t}$.

$$(1-\delta)w_i + \delta[(1-\delta)\min_{a_{I(\bar{t}-1)}} \max_{j \in E(i)\cap I(\bar{t}-1)} \max_{a'_j} u_i(\beta_{i,\bar{t}}(a_{I^b_{\bar{t}}}); a_{I(\bar{t}-1)}; a_{I^b_{\bar{t}-1}}) + \delta L_i]$$
$$\geq (1-\delta)w_i + \delta[(1-\delta)u_i(\beta_{i,\bar{t}-1}(a_{I^b_{\bar{t}-1}}); a_{I^b_{\bar{t}-1}}) + \delta u_i(\beta_{i,\bar{t}}(a_{I^b_{\bar{t}}}); a_{I^b_{\bar{t}}})]$$
$$\geq (1-\delta)w_i + \delta u_i(\beta_{i,\bar{t}-1}(a_{I^b_{\bar{t}-1}}); a_{I^b_{\bar{t}-1}}),$$

where the last inequality follows from Lemma 1. Generally, for any $a_{I^b_k}$ in step k of the given move pattern and for any $\delta \in (0,1)$, player i's continuation payoffs are not less than

$$(1-\delta^{\bar{t}-k})w_i + \delta^{\bar{t}-k}[(1-\delta)\min_{a_{I(k)}} \max_{j \in E(i)\cap I(k)} \max_{a'_j} u_i(\beta_{i,k+1}(a_{I(k)}; a_{I^b_k}); a_{I(k)}; a_{I^b_k}) + \delta L_i]$$
$$\geq (1-\delta^{\bar{t}-k})w_i + \delta^{\bar{t}-k}[(1-\delta)u_i(\beta_{i,k}(a_{I^b_k}); a_{I^b_k}) + \delta u_i(\beta_{i,\bar{t}}(a_{I^b_{\bar{t}}}); a_{I^b_{\bar{t}}})]$$
$$\geq (1-\delta^{\bar{t}-k})w_i + \delta^{\bar{t}-k}u_i(\beta_{i,k}(a_{I^b_k}); a_{I^b_k}),$$

where the last inequality follows from applying Lemma 1 repeatedly.

Therefore, player i's continuation payoffs in step 1 of the given move pattern are not less than $(1-\delta^{\bar{t}-1})w_i + \delta^{\bar{t}-1}\underline{r}^e_i(I(1), \ldots, I(\bar{t})) \geq (1-\delta^{\bar{t}-1})w_i + \delta^{\bar{t}-1}\underline{s}^e_i$ by the definition of the lower effective minimax value. Now, for any given $\epsilon > 0$, we get the desired result by taking $\underline{\delta}$ to satisfy $(1-\underline{\delta}^{\bar{t}-1})w_i + \underline{\delta}^{\bar{t}-1}\underline{s}^e_i > \underline{s}^e_i - \epsilon$. ■

Theorem 2 shows that any subgame perfect equilibrium payoff vector with correlation device in Γ essentially dominates the lower effective minimax point $(\underline{s}^e_1, \ldots, \underline{s}^e_n)$ of the repeated game Γ. This theorem, in particular, implies the result of Lagunoff and Matsui (1997) which shows that the only subgame perfect equilibrium payoff with correlation device in a single-move pure coordination game (when the players are sufficiently patient) is the one which Pareto-dominates all other payoffs, i.e., u^*. (See Proposition 1.) We next show that any payoff vector which strictly dominates the upper effective minimax point $(\underline{r}^e_1, \ldots, \underline{r}^e_n)$ is a subgame perfect equilibrium payoff vector with correlation device. We omit the proof since it is a minor modification of the corresponding proof in Yoon (2001): One needs only to pay extra attention to equivalent utilities.

Theorem 3. (Folk Theorem) *For any payoff vector $v \in V$ which strictly dominates the upper effective minimax point $(\underline{r}^e_1, \ldots, \underline{r}^e_n)$ of the repeated game $\Gamma = (G, \{I_t\}^{\bar{t}}_{t=1}, \delta)$, there exists a discount factor $\underline{\delta} < 1$ such that, for all $\delta \in (\underline{\delta}, 1)$, v is a subgame perfect equilibrium outcome with correlation device of Γ.*

Remark: This theorem is a characterization result. It is worthwhile to note that some games do not have a payoff vector that satisfies the theorem's assumption. See Example 3.

4. Conclusion

We have introduced two notions of effective minimax value for repeated games, the upper effective minimax value and the lower effective minimax value, in this paper. These values are in close relation with the standard and effective minimax values of the stage game: (i) When the NEU condition is satisfied, the upper effective minimax value coincides with the effective minimax value of the stage game (which in turn coincides with the standard minimax value), and (ii) When the repeated game is a simultaneous-move game, both the upper and the lower effective minimax values coincide with the effective minimax value of the stage game. Generally, however, asynchronous move structure affects both the upper and lower effective minimax values in significant ways.

The main results of the present paper established that these values constitute two cut-off levels of payoffs for the Folk Theorem: Any subgame perfect equilibrium payoff with correlation device dominates the lower effective minimax point, and any payoff that dominates the upper effective minimax point can be supported as a subgame perfect equilibrium outcome with correlation device.

We could not determine the *exact* cut-off level for the Folk Theorem, however. In other words, we were not able to establish which payoffs between the upper and lower effective minimax values can be supported as subgame perfect equilibrium outcomes with correlation device.[13] The exact cut-off level, if it could be satisfactorily characterized, would reflect the move patterns as well as the payoff structure of the stage game. We leave it as an agenda for future research, only noting that Example 5 may shed some light on it.[14] It is also worthwhile to extend the results to stochastic environments satisfying the FPIE condition, and to the unobservable mixed actions case.

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[13] Therefore, the lower bounds in Theorem 2 and 3 are not tight for some games.

[14] A recent paper by Wen (2002b) tries to tackle this issue.