The B.E. Journal of Economic Analysis & Policy

Contributions

Competitive Mixed Bundling of Vertically Differentiated Products

Illtae Ahn*   Kiho Yoon†

*Chung-Ang University, illtae@cau.ac.kr

†Korea University, kiho@korea.ac.kr

Competitive Mixed Bundling of Vertically Differentiated Products[*]

Illtae Ahn and Kiho Yoon

Abstract

We examine mixed bundling in a competitive environment that incorporates vertical product differentiation. We show that, compared to the equilibrium without bundling, (i) prices, profits and social welfare are lower, whereas (ii) consumer surplus is higher in the equilibrium with mixed bundling. In addition, the population of consumers who purchase both products from the same firm is larger in the equilibrium with mixed bundling. These results are largely in line with those obtained in the previous literature on competitive mixed bundling with horizontal differentiation. Further, we conduct a comparative static analysis with respect to changes in quality differentiation parameters. When the quality gap between brands narrows under no bundling and symmetric mixed bundling, prices and profits decrease. When quality differentiation is asymmetric across products, however, complicated effects occur on prices and profits due to strategic interdependence that mixed bundling creates.

KEYWORDS: mixed bundling, vertical differentiation, quality advantage, competitive bundling

1 Introduction

Mixed bundling, in which firms offer multiple products for sale individually or at a discount if purchased together, is commonly observed in the real world. Classic examples include stereo systems, software suites, fast-food restaurant meals, and concert tickets, which are typically available at a discount as packages alongside with individual products. Moreover, the strategic use of mixed bundling in oligopolistic environments is gaining increasing importance in the era of digital convergence. For example, many firms in the telecommunications and broadcasting industries offer the "triple play" service of combining telephony, Internet access, and broadcasting with bundle discounts to attract new customers and to widen their business sphere.[1]

In this paper, we examine mixed bundling in a competitive environment that incorporates vertical product differentiation. We compare the equilibrium prices, profits, social welfare and consumer surplus with mixed bundling to those without bundling to see the effects of mixed bundling. We also conduct a comparative static analysis with respect to changes in quality differentiation parameters.

There is an extensive literature on bundling.[2] Adams and Yellen (1976), Schmalensee (1982), McAfee, McMillan, and Whinston (1989), and Bakos and Brynjolfsson (1999) establish that the monopolist may use mixed bundling as a price discrimination device to increase profits. Other papers including Whinston (1990) and Nalebuff (2004) study how the monopolist makes strategic use of pure bundling or tying to foreclose competitors. These papers analyze the industry structure in which one firm holds a monopoly power or sufficient market power.

The strand of literature that is closest to the current paper is those that study mixed bundling in a more competitive and symmetric environment. These works include Matutes and Regibeau (1992), Anderson and Leruth (1993), Economides (1993), Reisinger (2006), Gans and King (2006), and Thanassoulis (2007).[3] In these papers, each of two symmetric and horizontally differentiated firms produces its own brand of two products.[4] Matutes and Regibeau (1992) study mixed bundling in the "mix and match" model of Matutes and Regibeau (1988) and show that, for a wide range of parameters, firms choose to offer bundle discounts even though their profits would be higher if they could agree not to be engaged in mixed bundling. They also show that firms' propensity to bundle is excessive from a social welfare perspective. Anderson and Leruth (1993) and

---

[1] Thanassoulis (2011) contains relevant discussion on bundling in multimedia markets.
[2] Excellent recent surveys include Kobayashi (2005) and Stole (2007).
[3] A recent paper by Armstrong and Vickers (2010) is also relevant.
[4] As a matter of fact, Gans and King (2006) first consider four independent firms and then integrated firms. See the next paragraph.

Economides (1993) consider mixed bundling in somewhat different models but arrive at basically the same conclusion.[5] Reisinger (2006) is yet another paper that studies mixed bundling in a circular-city model with a discussion of location choice. Thanassoulis (2007) extends Matutes and Regibeau's location model by introducing small buyers who only desire one of the component products and by distinguishing between firm-specific and product-specific preferences. This paper discovers that, compared to the equilibrium without bundling, profits are higher but consumer surplus is lower with mixed bundling under firm-specific preferences. The conclusions are reversed under product-specific preferences, with which the other papers are concerned.

In Gans and King (2006), two unrelated products are produced by four firms. Pairs of firms may agree to offer bundle discounts when their products are purchased together. In comparison to the papers above which assume that bundle discounts as well as stand-alone prices are set simultaneously, firms in this paper first decide on the level of bundle discounts and then simultaneously announce their stand-alone prices. Gans and King show that both pairs of firms offer bundle discounts such that all consumers purchase both products either from one pair or from the other in equilibrium. However, if both pairs of firms are integrated, they choose not to offer bundle discounts in equilibrium, contrary to the findings above.

No paper, however, that the authors are aware has studied mixed bundling in a competitive environment that incorporates vertical product differentiation. This is surprising since vertical differentiation in quality is widely observed in competitive mixed bundling. In fact, it would be an exception rather than the rule that the products offered by different firms have the same quality. Vertical differentiation may result from the product design or consumers' perception due to installed base, incumbency, or firm's reputation. As an example, consider the broadcasting and telecommunications industry. Both the phone company and the cable company offer the telephony service and the broadcasting service. In many instances, the phone company offers a high-quality telephony service, whereas the cable company offers a high-quality broadcasting service. The telecommunications industry itself in many countries, say in Korea, is another example of a vertically differentiated oligopoly engaged in mixed bundling. Korea Telecom (KT) has market dominance in fixed-line telecommunications, whereas SK Telecom (SKT) holds the dominant position in mobile telecommunications. In recent years, each firm has expanded into the other's turf by merger and acquisitions, and started to offer bundles alongside with stand-alone telephony services. Consumers' general perception is such that KT's quality of fixed-line telephony service is superior, but SKT's mobile telephony service is superior.

---

[5] Matutes and Reibeau (1992) adopts the Hotelling location model; Anderson and Leruth (1993), the "discrete choice" model; and Economides (1993), the linear demand model.

To analyze the effects of competitive mixed bundling in vertically differentiated product markets, we consider a model in which two firms offer their own brands of two products. One firm offers a high-quality brand of product 1 and a low-quality brand of product 2, whereas the other offers a low-quality brand of product 1 and a high-quality brand of product 2. We assume that (i) there are no economies of scope at the production level, i.e., bundling does not induce cost savings,[6] and that (ii) consumers' valuations for the two products are independent. Hence, we rule out the rationale for bundling arising out of production and/or consumption efficiency.

After introducing the model in Section 2, we consider the benchmark case in which the firms do not offer bundle discounts in Section 3. We then analyze the case of mixed bundling in Section 4: We start in Section 4.1 with the case when the firms' quality advantages are symmetric. In Section 4.2, we allow for asymmetric quality advantages. That is, the magnitude of the quality gap between the brands of product 1 may be different from that between the brands of product 2. It is shown that, compared to the equilibrium without bundling, (i) prices, profits and social welfare are lower, whereas (ii) consumer surplus is higher in the equilibrium with mixed bundling. In addition, the population of consumers who purchase both products from the same firm is larger in the equilibrium with mixed bundling.

These results are in line with those obtained in the previous literature on competitive mixed bundling with horizontally differentiated products. Mixed bundling, both with horizontal differentiation and with vertical differentiation, intensifies competition between the firms by extending it to the inter-products level, as the bundles become substitutes. This lowers prices and profits. Social welfare decreases in the horizontal differentiation models because some consumers who like one product more than the other and thus would be best served by purchasing from different firms are lured into buying bundles as a result of bundle discounts. The decrease in social welfare in our model can be explained in a similar way: some consumers who care about the qualities of both products switch from buying high-quality brands of both products to buying bundles. However, mixed bundling in our model has additional effect. It increases social welfare by shrinking or eliminating the consumer group who purchase low-quality brands of both products. Social welfare decreases because the former effect dominates the latter.

We conduct a comparative static analysis with respect to changes in quality differentiation parameters. Let A be the quality gap between the brands of one product and B be the quality gap between the brands of another product. We

---

[6] In fact, marginal production costs will be assumed to be identical for all quality brands and for all firms.

show in Section 3 that, when firms do not offer bundle discounts, equilibrium prices and profits decrease as A or B gets smaller. The reason is basically that firms compete more intensely when brands become more substitutable. We show in Section 4.1 for symmetric mixed bundling that, with respect to a decrease in $A = B$, the direction of changes in these equilibrium variables is the same as that in the previous section but the rate of changes is different. In particular, prices and profits decrease more slowly as $A = B$ decreases. Hence, the profit-reducing effect of mixed bundling gets exacerbated as the quality gap, $A = B$, increases.

The asymmetric quality differentiation in Section 4.2 yields more interesting observations with regard to the comparative statics. Let us assume without loss of generality that $A \geq B$. We first show that a decrease in A reduces both firms' profits, whereas a decrease in B increases one firm's profits but reduces the other's profits. The reason is that a decrease in A narrows the quality gap between bundles and thus intensifies competition. In contrast, a decrease in B widens the quality gap which leads to lessened competition, favoring the firm in a competitively advantageous position. The comparative statics on the profit-reducing effect of mixed bundling become slightly different from the case of symmetric mixed bundling. We also show that the effect of a decrease in A on prices is different from that of a decrease in B, due to rather complicated strategic interdependence between the products.[7] We additionally establish that the equilibrium prices with mixed bundling converge to those without bundling when the quality gap in one product vanishes, that is, when B gets close to zero.

In Section 5, we extend the model by relaxing some of the assumptions and show that the main results continue to hold with these extensions. Section 6 concludes.

2   The model

Two firms, *N* and *S*, each offer their own brand of two products, 1 and 2. The products are differentiated in terms of their quality. That is, firm *N* offers a high-quality brand of product 1 and a low-quality brand of product 2, and vice versa. We assume that the marginal cost of production is zero for both firms.[8] Let p_i^k denote the (stand-alone) price charged by firm $k = N, S$ for product $i = 1, 2$ and δ^k denote the bundle discount offered by firm $k = N, S$ when the two products are sold as a package. We assume that $p_i^k \geq 0$ and $\delta^k \geq 0$. The firms set their prices *and* bundle discounts simultaneously.

---

[7]   Please refer to Section 4.2 for more details.
[8]   The main results of the present paper continue to hold even when we relax this assumption of zero marginal costs. See Section 5 for more details.

There is a population of consumers who purchase at most one unit of each of the two products. We normalize the total population of consumers to be 1. Consumers differ in their "taste for quality." A consumer's preference is denoted by $(x, y) \in [0,1] \times [0,1]$, where x represents the valuation for product 1 and y represents that for product 2. We assume that consumers' valuations for the products are uniformly distributed on a unit square $[0,1] \times [0,1]$. Hence, consumers' preferences for the two products are independent. A consumer with the preference (x, y) enjoys a gross payoff of $v_1 + a^k x$ when she purchases one unit of product 1 from firm $k = N, S$, and she enjoys a gross payoff of $v_2 + b^k y$ when she purchases one unit of product 2 from firm $k = N, S$. Observe that this payoff specification is equivalent to the one in the seminal work of Shaked and Sutton (1983) and the subsequent works on vertical differentiation. Up to Section 4, we assume that v_1 and v_2 are sufficiently large such that every consumer purchases one unit of each of the two products. (We relax this *full market coverage* assumption in Section 5.) Hence, a consumer may purchase (i) one unit of product 1 from *N* and one unit of product 2 from *S*; (ii) one unit of each of the two products from *N*; (iii) one unit of each of the two products from *S*; or (iv) one unit of product 1 from *S* and one unit of product 2 from *N*. Accordingly, a consumer with the preference (x, y) enjoys a net payoff as follows:

(i) $v_1 + v_2 + a^N x + b^S y - (p_1^N + p_2^S)$,

(ii) $v_1 + v_2 + a^N x + b^N y - (p_1^N + p_2^N - \delta^N)$,

(iii) $v_1 + v_2 + a^S x + b^S y - (p_1^S + p_2^S - \delta^S)$, or

(iv) $v_1 + v_2 + a^S x + b^N y - (p_1^S + p_2^N)$,

where a^N and b^N are firm N's quality parameters for products 1 and 2, respectively, and a^S and b^S are firm S's quality parameters for products 1 and 2, respectively. Here, $a^N > a^S$, but $b^N < b^S$, that is, firm N has a quality advantage in product 1, whereas firm S has a quality advantage in product 2. For simplicity, we fix the qualities of the high-quality brands of both products to 1. Thus,

Assumption. $a^N = b^S = 1$ and $0 < a^S, b^N < 1$.

We make several notational conventions. Firstly, we denote the parameters a^S and b^N as a and b, respectively. Secondly, we denote the bundle price $p_1^k + p_2^k - \delta^k$ as r^k for $k = N, S$. Then, a consumer's net payoff for the four

purchase scenarios discussed above, as suppressing the common component $v_1 + v_2$, can be rewritten as follows:

(i) $x + y - p_1^N - p_2^S$,
(ii) $x + by - r^N$,
(iii) $ax + y - r^S$, or
(iv) $ax + by - p_1^S - p_2^N$.

Let us denote the population of consumers who purchase product 1 from firm N and product 2 from firm S by D^{NS}, the population of consumers who purchase both products from firm N by D^{NN}, and so on.

3   Equilibrium without bundling

As a benchmark, we first consider the case when no firm offers a bundle discount, that is, $\delta^N = \delta^S = 0$. Figure 1 shows the consumer's choices without bundling. Consumers with $x \geq (p_1^N - p_1^S)/(1-a)$ purchase product 1 from firm N, whereas consumers with $x < (p_1^N - p_1^S)/(1-a)$ purchase product 1 from firm S. Similarly, consumers with $y \geq (p_2^S - p_2^N)/(1-b)$ purchase product 2 from firm S, whereas consumers with $y < (p_2^S - p_2^N)/(1-b)$ purchase product 2 from firm N.
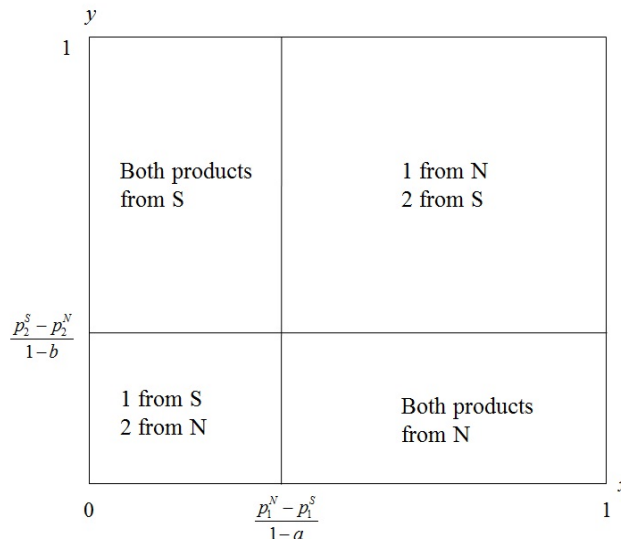

Figure 1: The pattern of consumer choice without bundling

Hence, the firms' profits are

$$\pi^N = p_1^N(1 - \frac{p_1^N - p_1^S}{1-a}) + p_2^N \frac{p_2^S - p_2^N}{1-b}; \pi^S = p_1^S \frac{p_1^N - p_1^S}{1-a} + p_2^S(1 - \frac{p_2^S - p_2^N}{1-b}).$$

We have:

Proposition 1. *The equilibrium is given by*

$$p_1^N = \frac{2(1-a)}{3}, p_1^S = \frac{1-a}{3}, p_2^N = \frac{1-b}{3}, p_2^S = \frac{2(1-b)}{3};$$

$$D^{NS} = \frac{4}{9}, D^{NN} = D^{SS} = \frac{2}{9}, D^{SN} = \frac{1}{9}; \pi^N = \frac{5-4a-b}{9}, \pi^S = \frac{5-a-4b}{9}.$$

Proof. The equilibrium can be easily obtained by setting $\partial \pi^k / \partial p_i^k = 0$ for $i = 1, 2$ and for $k = N, S$.   Q.E.D.

  Observe that the equilibrium prices p_i^N and p_i^S of product i depend only on the quality differentiation parameter for product i, not on that for the other product. In this sense, competition between the firms is completely separated across the products, that is, there is no strategic interdependence between the products. This aspect of competition changes when the firms offer bundle discounts. Observe also that when the quality differentiation between the firms diminishes (i.e., as a or b increases toward 1), both prices and profits decrease. The reason is that the firms compete more vigorously as the products become more substitutable.
  Social welfare without bundling is given by

$$\int_0^{1/3} axdx + \int_{1/3}^1 xdx + \int_0^{1/3} bydy + \int_{1/3}^1 ydy = \frac{16+a+b}{18}.$$

Consumer surplus, which is equal to social welfare net of the firms' profits, is $(11a + 11b - 4)/18$.[9] Both social welfare and consumer surplus increase as a or b increases.

---

[9] Note that we suppress the common component $v_1 + v_2$.

4 Equilibrium with mixed bundling

We now assume that the firms are engaged in mixed bundling. Then the consumer's choice can be summarized by the following two diagrams. Figure 2 depicts the case when $\min\{r^N, r^S\} > p_1^S + p_2^N$, whereas Figure 3 depicts the case when $\min\{r^N, r^S\} \le p_1^S + p_2^N$.

Figure 2 shows that consumers with high valuations for both products purchase product 1 from firm N and product 2 from firm S. Consumers with a high valuation for product 1 but a low valuation for product 2 purchase both product from N, whereas those with a high valuation for product 2 but a low valuation for product 1 purchase both product from S. Finally, consumers with low valuations for both products purchase product 1 from S and product 2 from N.
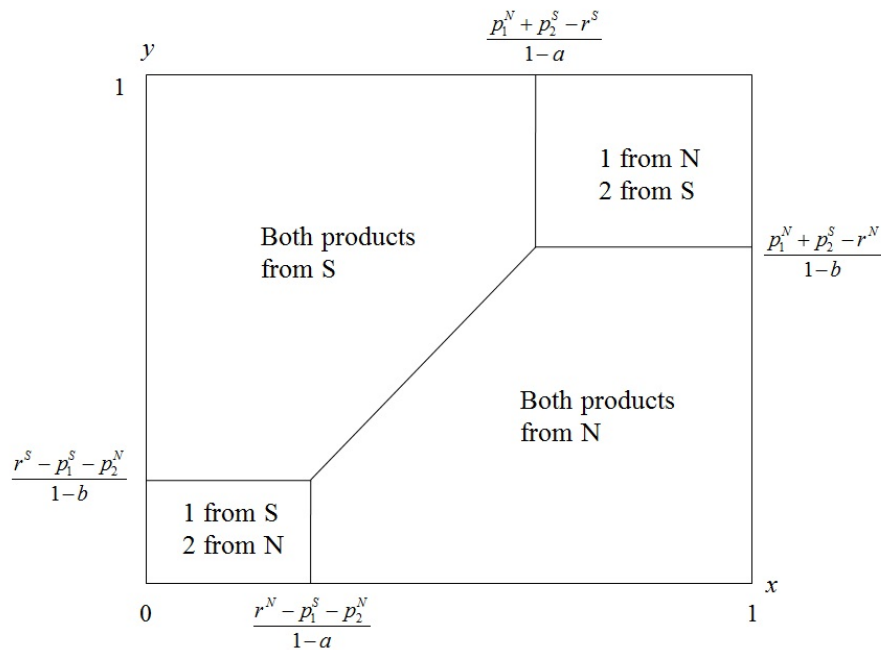
Figure 2: The pattern of consumer choice when $\min\{r^N, r^S\} > p_1^S + p_2^N$

Figure 3 shows similar purchase patterns, except that there do not exist consumers who purchase product 1 from S and product 2 from N. Figure 3(a) shows the case when $r^S \le r^N$, and Figure 3(b) shows the case when $r^S > r^N$. If a consumer purchased product 1 from S and product 2 from N, then she would consume low-quality brands of both products. When $\min\{r^N, r^S\} = r^S \le p_1^S + p_2^N$,

she can consume a high-quality brand of product 2 (and a low-quality brand of product 1) with a lower total price by purchasing both products from S. Similarly, when $\min\{r^N, r^S\} = r^N \leq p_1^S + p_2^N$, she can consume a high-quality brand of product 1 (and a low-quality brand of product 2) with a lower total price by purchasing both products from N.

(a)   (b)
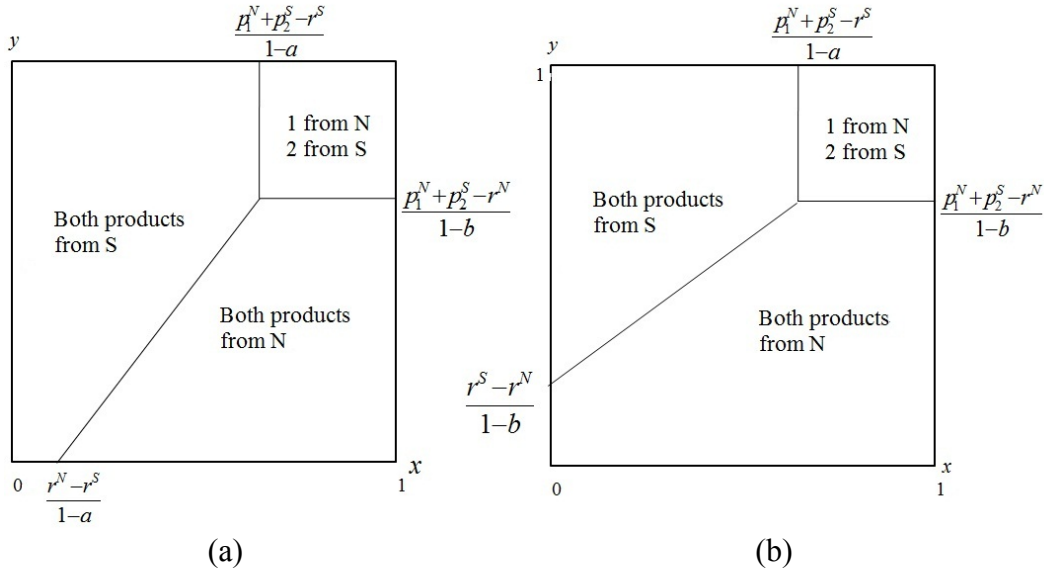
Figure 3: The pattern of consumer choice when $\min\{r^N, r^S\} \leq p_1^S + p_2^N$

The firms' profits when $\min\{r^N, r^S\} = r^S \leq p_1^S + p_2^N$ are

$$\pi^N = p_1^N D^{NS} + r^N D^{NN} = p_1^N \left(1 - \frac{p_1^N + p_2^S - r^S}{1-a}\right)\left(1 - \frac{p_1^N + p_2^S - r^N}{1-b}\right)$$

$$+ r^N \left\{ \frac{p_1^N + p_2^S - r^N}{1-b}\left(1 - \frac{p_1^N + p_2^S - r^S}{1-a}\right) \right.$$

$$\left. + \frac{1}{2}\frac{p_1^N + p_2^S - r^N}{1-b}\left(\frac{p_1^N + p_2^S - r^S}{1-a} - \frac{r^N - r^S}{1-a}\right)\right\},$$

$$\pi^S = p_2^S D^{NS} + r^S D^{SS} = p_2^S \left(1 - \frac{p_1^N + p_2^S - r^S}{1-a}\right)\left(1 - \frac{p_1^N + p_2^S - r^N}{1-b}\right)$$

$$+ r^S \left\{ \frac{p_1^N + p_2^S - r^S}{1-a}\left(1 - \frac{p_1^N + p_2^S - r^N}{1-b}\right)\right.$$

$$\left. + \frac{1}{2}\frac{p_1^N + p_2^S - r^N}{1-b}\left(\frac{p_1^N + p_2^S - r^S}{1-a} + \frac{r^N - r^S}{1-a}\right)\right\},$$

and their profits when $\min\{r^N, r^S\} = r^N \le p_1^S + p_2^N$ can be analogously given. We can in fact establish that there cannot exist consumers who purchase low-quality brands of both products. That is, the situation in Figure 2 cannot happen in equilibrium.

Proposition 2. *We have* $\min\{r^N, r^S\} \le p_1^S + p_2^N$ *in equilibrium.*

Proof. See Appendix.   Q.E.D.

The reason for this result is that, when $\min\{r^N, r^S\} > p_1^S + p_2^N$ holds, either firm N can profitably increase p_2^N to $\min\{r^N, r^S\} - p_1^S$ or firm S can profitably increase p_1^S to $\min\{r^N, r^S\} - p_2^N$ such that there exist no consumers who purchase low-quality brands of both products. By raising the prices of low-quality brands to make the option of buying low-quality brands of both products unattractive, the firms induce consumers to purchase bundles.[10]

Observe that we have drawn Figure 3 under the presumption that $(p_1^N + p_2^S - r^S)/(1-a) < 1$ and $(p_1^N + p_2^S - r^N)/(1-b) < 1$. If one (or both) of these inequalities does not hold, then there exist no consumers who purchase product 1 from firm N and product 2 from firm S. That is, all consumers purchase either N's bundle or S's bundle. However, we establish in Proposition 3 that this pure bundling outcome cannot happen in equilibrium. In a situation where all consumers purchase either firm's bundle, we can show that one firm can profitably reduce the stand-alone price of its strong product to capture some consumers who have high valuations for both products but used to buy the bundle of the other firm.

---

[10] We note that Proposition 2 holds under the full market coverage assumption. When the market is not fully covered, the deviation used in proving the proposition (raising the prices of low-quality brands) would result in reduced demand, with potentially ambiguous effects on profits. In fact, we show in Section 5 that $D^{SN} > 0$ in equilibrium if the full market coverage assumption is dropped. The authors are indebted to an anonymous referee for pointing out this issue.

Proposition 3. *We have* $\max\left\{\dfrac{p_1^N + p_2^S - r^S}{1-a}, \dfrac{p_1^N + p_2^S - r^N}{1-b}\right\} < 1$ *in equilibrium.*

Proof. See Appendix.   Q.E.D.

We next establish that $r^S \leq r^N$ when $a \leq b$. That is, if firm N's disadvantage in product 2 is smaller than firm S's disadvantage in product 1, so that the quality of firm N's bundle is superior to that of firm S's bundle, then firm N's bundle price is higher in equilibrium.

Proposition 4. *Assume that* $a \leq b$. *We have* $r^S \leq r^N$ *in equilibrium.*

Proof. See Appendix.   Q.E.D.

4.1   Mixed bundling under symmetric quality differentiation

We now focus on the case in which the firms' quality disadvantages are symmetric, that is, $a = b$. Section 4.2 below addresses a more general case in which $a \neq b$. The following proposition characterizes the equilibrium outcome.

Proposition 5. *Assume that* $a = b$. *The equilibrium is given by*

$$p_1^N = p_2^S = \frac{7(1-a)}{12}, r^N = r^S = \frac{2(1-a)}{3};$$

$$D^{NS} = \frac{1}{4}, D^{NN} = D^{SS} = \frac{3}{8}, D^{SN} = 0; \pi^N = \pi^S = \frac{19(1-a)}{48}.$$

Proof. See Appendix.   Q.E.D.

Let us compare the equilibrium outcome with mixed bundling to that without bundling. Since we are now focusing on the symmetric case for the moment, we need to set $a = b$ in Proposition 1. Observe first that the population of consumers who purchase both products from the same firm, i.e., $D^{NN} + D^{SS}$, increases from 4/9 without bundling to 3/4 with mixed bundling. This is because the firms compete to attract customers by offering bundle discounts. Hence, the bundle price $r^N = r^S = 2(1-a)/3$ is lower than the sum of stand-alone prices without bundling, which is equal to $1-a$. Note also that the stand-alone prices of high-quality brands, $p_1^N = p_2^S$, are lower with mixed bundling. As a consequence,

the firms' profits also decrease from $5(1-a)/9$ without bundling to $19(1-a)/48$ with mixed bundling.[11] Most of the papers on competitive bundling with horizontal differentiated products, including Matutes and Regibeau (1992), Anderson and Leruth (1993), and Economides (1993), show that the firms' profits decrease in the equilibrium with mixed bundling compared to the equilibrium without bundling. We have established that this finding continues to hold when products are vertically differentiated. As a matter of fact, basically the same intuition applies. The reason why mixed bundling reduces profits in our model is that it creates a new aspect of competition. Recall from the discussion after Proposition 1 that competition between the firms is completely separated across the products without bundling. With mixed bundling, the competition between firms is extended to the inter-product level as the bundles offered by the two firms become substitutes. Mixed bundling in the horizontal differentiation models has the exact same role.[12]

Social welfare with mixed bundling is given by

$$\int_{1/2}^{1}\int_{1/2}^{1}(x+y)dydx + \int_{1/2}^{1}\int_{0}^{1/2}(x+by)dydx + \int_{0}^{1/2}\int_{0}^{x}(x+by)dydx$$
$$+\int_{0}^{1/2}\int_{1/2}^{1}(ax+y)dydx + \int_{0}^{1/2}\int_{x}^{1/2}(ax+y)dydx = \frac{5+a}{6}.$$

Recall from the previous section that social welfare without bundling is $(8+a)/9$ when $a=b$. Since $(8+a)/9 - (5+a)/6 = (1-a)/18 > 0$, social welfare decreases with mixed bundling. Mixed bundling has two countervailing effects on social welfare. On the one hand, it increases social welfare by eliminating the consumer group, represented by D^{SN}, who purchase low-quality brands of both products. On the other hand, it decreases social welfare by shrinking the consumer group, represented by D^{NS}, who purchase high-quality brands of both products. As can be seen from Figures 1 and 3 (with setting $a=b$ and $r^{N} = r^{S}$), the former effect increases social welfare by

---

[11] We can show, from the first order conditions given in the appendix, that one firm's best response is to offer mixed bundling when the other does not. For instance, when $a = b = 0$, firm N's best response is to set $r^{N} = 0.81083$ and $p_1^{N} = 0.72735$ if firm S follows the equilibrium strategy without bundling, i.e. $p_1^{S} = 1/3$, $p_2^{S} = 2/3$, and $r^{S} = p_1^{S} + p_2^{S} = 1$. Thus, firms face a prisoners' dilemma situation, similar to the case of mixed bundling with horizontal differentiation.

[12] Note that, though the individual products are vertically differentiated in our model, the bundles are horizontally differentiated, especially when $a = b$. Some consumers prefer firm N's bundle while others prefer firm S's bundle, depending on their preference for the individual components of the bundles. We appreciate an anonymous referee for pointing this out.

$$\int_0^{1/3} \int_0^x [x+by-(ax+by)]\,dydx + \int_0^{1/3} \int_0^y [ax+y-(ax+by)]\,dxdy$$

$$= 2\int_0^{1/3} \int_0^x (1-a)x\,dydx = \frac{2(1-a)}{81},$$

whereas the latter effect decreases social welfare by

$$\int_{1/2}^1 \int_{1/3}^{1/2} [x+y-(x+by)]\,dydx + \int_{1/3}^{1/2} \int_{1/3}^x [x+y-(x+by)]\,dydx$$

$$+ \int_{1/2}^1 \int_{1/3}^{1/2} [x+y-(ax+y)]\,dxdy + \int_{1/3}^{1/2} \int_{1/3}^y [x+y-(ax+y)]\,dxdy$$

$$= 2[\int_{1/2}^1 \int_{1/3}^{1/2} (1-b)y\,dydx + \int_{1/3}^{1/2} \int_{1/3}^x (1-b)y\,dydx]$$

$$= \frac{13(1-a)}{162} \quad .$$

Thus, the latter effect dominates. It is clear that firms offer bundle discounts mainly to attract the consumers in D^{NS} who have high valuations for both products. The extinction of consumer group D^{SN} may be viewed as a by-product of firms' competition at this front. In other words, the latter effect is primary whereas the former effect is auxiliary.[13] That's why social welfare decreases with mixed bundling.

In existing models of competitive mixed bundling with horizontal differentiation, such as Matutes and Regibeau (1992), Gans and King (2006), and Reisinger (2006), social welfare decreases with mixed bundling due to the distributive inefficiency. That is, social welfare decreases in these location models because some consumers who like one product more than the other and thus would be best served by purchasing from different firms are lured into buying bundles as a result of bundle discounts. In this respect, the reason for the decrease in social welfare is similar to the case of vertical differentiation where some consumers who care about the qualities of both products switch from buying high-quality brands of both products to buying bundles. Note, however, that the equilibrium without bundling in horizontal differentiation models is socially

---

[13] Let us elaborate on this. The dominance of the latter effect can be explained by two reasons. First of all, mixed bundling shrinks the size of D^{NS} more than that of D^{SN}. Observe that the size of D^{NS} used to be larger than that of D^{SN} without bundling, due to high-quality brands' quality advantages (and the assumption of uniform distribution). Thus, mixed bundling affects the size of D^{NS} more than that of D^{SN}. Secondly, the consumers who used to be in D^{NS} without bundling care much more about the quality than those in D^{SN}. Hence, a change in their consumption has a bigger impact on social welfare.

optimal since the total distance 'travelled' by consumers is minimized whereas the corresponding equilibrium in this vertical differentiation model is not. The social optimum is achieved when all consumers purchase high-quality brands from both firms. Thus, it is less obvious that social welfare decreases with mixed bundling in our model.

Finally, consumer surplus with mixed bundling is $(1+23a)/24$. Thus, consumer surplus increases with mixed bundling by the amount of $19(1-a)/72$. This is mainly due to price decreases under mixed bundling. As a matter of fact, it is straightforward to check that all consumers, including those who change their consumption patterns, are better off under mixed bundling.

Summarizing the discussion,

Proposition 6. *Compared to the equilibrium without bundling, (i) prices, profits and social welfare are lower, whereas (ii) consumer surplus as well as the population of consumers who purchase both products from the same firm is larger in the equilibrium with mixed bundling.*

As for comparative statics, observe in Proposition 5 that prices as well as profits decrease as $a = b$ increases. Observe also from the discussion above that both social welfare and consumer surplus increase as $a = b$ increases.

A noteworthy observation with respect to the effect of a change in the parameter $a = b$ is that the profit-reducing effect of mixed bundling gets exacerbated as the degree of vertical differentiation increases. In particular, the magnitude of reduction in profits as the firms offer bundle discounts, which is equal to $5(1-a)/9 - 19(1-a)/48 = 23(1-a)/144$, increases with higher vertical differentiation, i.e., with a decrease in $a = b$. As explained before, competition is intensified with mixed bundling since the bundles become substitutes. This competition-enhancing effect of mixed bundling, which certainly reduces profits, is stronger when the competition level without bundling is lower. On the other hand, when the competition level is already high even without bundling, as in the case when $a = b$ is sufficiently close to 1, firms do not have much room for bundle discounts. Therefore, the profit-reducing effect of mixed bundling becomes bigger when the quality gap increases.

4.2 Mixed bundling under asymmetric quality differentiation

Let us return to the general case in which $a \neq b$. Assume without loss of generality that $a \leq b$. We relegate the characterization of equilibrium to the appendix since it is technical as well as quite complicated. The technical lemma in Appendix shows that the equilibrium prices with mixed bundling depend on the

quality differentiation parameters a and b for both products. This result contrasts with that for the case without bundling, where the equilibrium prices of product i depend only on the quality differentiation parameter for product i, not on that for the other product. A detailed discussion on this strategic interdependence will follow shortly.

Let us compare the equilibrium outcome with mixed bundling to that without bundling. In short, the result is the same as that for the symmetric case. We show in Proposition 7 that the equilibrium prices are lower and the population of consumers who purchase bundles is larger with mixed bundling.

Proposition 7. *Assume that $a \leq b$. Compared to the equilibrium without bundling, (i) prices are lower, whereas (ii) the population of consumers who purchase both products from the same firm is larger in the equilibrium with mixed bundling.*

Proof. See Appendix.   Q.E.D.

We can obtain the same results for profits, social welfare, and consumer surplus as we have in Proposition 6. However, since the proof is tediously long albeit similar to that of Proposition 7, we only present some numerical results.[14] In particular, we present the results for profits in Tables 1 and 2 below, but relegate the results on social welfare and consumer surplus to Tables 5 and 6 in the appendix for expositional brevity. Comparing these values with the corresponding equilibrium outcomes without bundling (obtained analytically at Proposition 1 and whose numerical values are given in the parentheses at the tables below), we can easily see that profits and social welfare are lower but consumer surplus is higher under mixed bundling.

---

[14] The proof is available upon request.

Table 1 π^N under mixed bundling

a \ b	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.3958	0.3979	0.4006	0.4038	0.4076	0.4121	0.4173	0.4231	0.4296	0.4367
	(0.5556)	(0.5444)	(0.5333)	(0.5222)	(0.5111)	(0.5)	(0.4889)	(0.4778)	(0.4667)	(0.4556)
0.1		0.3563	0.3584	0.3611	0.3645	0.3686	0.3734	0.3790	0.3853	0.3923
		(0.5)	(0.4889)	(0.4778)	(0.4667)	(0.4556)	(0.4444)	(0.4333)	(0.4222)	(0.4111)
0.2			0.3167	0.3189	0.3217	0.3253	0.3297	0.3349	0.3410	0.3479
			(0.4444)	(0.4333)	(0.4222)	(0.4111)	(0.4)	(0.3889)	(0.3778)	(0.3667)
0.3				0.2771	0.2793	0.2823	0.2862	0.2910	0.2910	0.3035
				(0.3889)	(0.3778)	(0.3667)	(0.3556)	(0.3444)	(0.3333)	(0.3222)
0.4					0.2375	0.2398	0.2430	0.2473	0.2527	0.2591
					(0.3333)	(0.3222)	(0.3111)	(0.3)	(0.2889)	(0.2778)
0.5						0.1979	0.2003	0.2038	0.2086	0.2148
						(0.2778)	(0.2667)	(0.2556)	(0.2444)	(0.2333)
0.6							0.1583	0.1609	0.1649	0.1705
							(0.2222)	(0.2111)	(0.2)	(0.1889)
0.7								0.1188	0.1215	0.1263
								(0.1667)	(0.1556)	(0.1444)
0.8									0.0792	0.0824
									(0.1111)	(0.1)
0.9										0.0396
										(0.0556)

Table 2 π^S under mixed bundling

b / a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.3958 (0.5556)	0.3676 (0.5111)	0.3391 (0.4667)	0.3106 (0.4222)	0.2819 (0.3778)	0.2532 (0.3333)	0.2245 (0.2889)	0.1959 (0.2444)	0.1675 (0.2)	0.1392 (0.1556)
0.1		0.3563 (0.5)	0.3280 (0.4556)	0.2995 (0.4111)	0.2709 (0.3667)	0.2422 (0.3222)	0.2135 (0.2778)	0.1849 (0.2333)	0.1564 (0.1889)	0.1281 (0.1444)
0.2			0.3167 (0.4444)	0.2884 (0.4)	0.2599 (0.3556)	0.2312 (0.3111)	0.2025 (0.2667)	0.1739 (0.2222)	0.1454 (0.1778)	0.1170 (0.1333)
0.3				0.2771 (0.3889)	0.2488 (0.3444)	0.2203 (0.3)	0.1916 (0.2556)	0.1629 (0.2111)	0.1343 (0.1667)	0.1059 (0.1222)
0.4					0.2375 (0.3333)	0.2092 (0.2889)	0.1806 (0.2444)	0.1519 (0.2)	0.1233 (0.1556)	0.0948 (0.1111)
0.5						0.1979 (0.2778)	0.1696 (0.2333)	0.1409 (0.1889)	0.1123 (0.1444)	0.0837 (0.1)
0.6							0.1583 (0.2222)	0.1299 (0.1778)	0.1013 (0.1333)	0.0727 (0.0889)
0.7								0.1188 (0.1667)	0.0903 (0.1222)	0.0616 (0.0778)
0.8									0.0792 (0.1111)	0.0506 (0.0667)
0.9										0.0396 (0.0556)

We now turn to the comparative static analysis. Observe from Tables 1 and 2 that an increase in a reduces both firms' profits, whereas an increase in b increases firm N's profits but reduces firm S's profits. This asymmetry is due to our assumption of $a \leq b$, or more accurately, it is due to the difference in the effect of changes in quality parameters on the firms' competitive advantage: An increase in a, which means a reduction in firm S's quality disadvantage in product 1, diminishes firm N's overall advantage. On the other hand, an increase in b or a reduction in firm N's quality disadvantage in product 2 enhances firm N's overall advantage. In other words, an increase in a narrows the quality gap between the firms' bundles, whereas an increase in b widens it. This contrasts with the result in Section 3 that an increase in a or b reduces both firms' profits in the case without bundling, which is due to no strategic interdependence across the products. Observe also from Tables 5 and 6 in the appendix that both social welfare and consumer surplus increase in a and b.

As we have seen in the case of $a = b$, the profit-reducing effect of mixed bundling gets exacerbated as $a = b$ decreases. When $a < b$, we have a slightly different result. When b decreases, we have the same result as in the case of

$a = b$. We can verify from Tables 1, 2 and Proposition 1 that the reduction in both firms' profits gets larger as b decreases. A decrease in a also makes the reduction in firm N's profits larger. However, the reduction in firm S's profits turns out to be smaller for some parameter values as a decreases. How can we explain this difference? Recall from the case of $a = b$ that firms do not have much room for bundle discounts when the quality gap is small. This intuition also applies when $a < b$. When a or b (separately) decreases, both firms' affordability to provide bundle discounts increases as their profits are higher without bundling. Therefore, a decrease in a or b has an effect to exacerbate the profit-reducing effect of mixed bundling. However, there are additional aspects to be taken into account when $a < b$. As explained in the last paragraph, an increase in a narrows the quality gap between the firms' bundles, whereas an increase in b widens it. Thus, a decrease in b makes the profit-reducing effect of mixed bundling larger, while a decrease in a tends to make it smaller. Therefore, a decrease in a has an ambiguous overall effect on the profit-reducing effect of mixed bundling. On the contrary, a decrease in b unambiguously makes the profit-reducing effect of mixed bundling larger.

The effect of changes in quality parameters on the equilibrium prices is more intricate. (We relegate the numerical results for prices to Tables 7, 8, 9 and 10 in the appendix.) An increase in a lowers p_1^N, r^N, and r^S, whereas its effect on p_2^S is ambiguous.[15] This is due to strategic interdependences between the products that mixed bundling creates. Without bundling, product 1 and product 2 are neither substitutes nor complements. The demand for product 1 offered by firm N (or S) is independent of the price of product 2 offered by firm S (or N), and vice versa. Mixed bundling changes this relationship. Figure 3(a) reveals the following strategic interdependences: (i) the bundles offered by the two firms are substitutes; (ii) product 1 by firm N and the bundle by firm S are substitutes, and so are product 2 by firm S and the bundle by firm N; (iii) product 1 and the bundle by firm N are substitutes, and so are product 2 and the bundle by firm S; and (iv) product 1 by firm N and product 2 by firm S are complements.[16] Taking these interdependences into account, suppose that the quality of product 1 by firm S improves, that is, the parameter a increases. As the rival firm's disadvantage in product 1 diminishes, firm N will lower both the

---

[15] See, for instance, the case of $b = 0.8$ or $b = 0.9$ in Table 8 in the appendix.

[16] The bundles offered by the firms are substitutes in the sense that the demand for firm N's bundle, D^{NN}, is increasing in the price of firm S's bundle, r^S, and vice versa. On the other hand, product 1 by firm N and product 2 by firm S are complements, because the demand for both, which is equal to D^{NS}, is decreasing in p_1^N and p_2^S.

bundle price r^N and stand-alone price p_1^N of product 1. The bundle price r^S also decreases by (i) and (ii). However, the effect on p_2^S is ambiguous because a decrease in r^N and r^S lowers p_2^S by (ii) and (iii) but a decrease in p_1^N raises p_2^S by (iv).

The effect of an increase in b is slightly different. An increase in b lowers both p_2^S and r^S by a reason similar to the one given above. On the other hand, Table 9 shows that the effect on r^N is ambiguous.[17] This is because an increase in b, by improving the quality of firm N's bundle, has an effect of pushing up r^N. Recall that an increase in a also has an effect of raising r^S. But this effect is smaller than the effect on r^N of an increase in b since an increase in a narrows the quality gap between the firms' bundles whereas an increase in b widens it. Thus, an increase in a has a negative overall effect on r^S, whereas an increase b has an ambiguous overall effect on r^N. Because of these differential effects on the quality gap, an increase in b unambiguously raises p_1^N, which contrasts with the fact that an increase in a has an ambiguous effect on p_2^S.

The following table summarizes the comparative static results for the mixed bundling case. In the table, SW denotes social welfare and CS denotes consumer surplus. In addition, $\Delta\pi^k$ for $k = N, S$ denotes the difference in firm k's profits under no bundling and under mixed bundling.

Table 3   Comparative statics for the mixed bundling case

	∂p_1^N	∂p_2^S	∂r^N	∂r^S	$\partial\pi^N$	$\partial\pi^S$	∂SW	∂CS	$\partial\Delta\pi^N$	$\partial\Delta\pi^S$
symmetric quality differentiation ($a = b$)										
$\partial a = \partial b$	−	−	−	−	−	−	+	+	−	−
asymmetric quality differentiation ($a \leq b$)										
∂a	−	±	−	−	−	−	+	+	−	±
∂b	+	−	±	−	+	−	+	+	−	−

We finally establish that, as b gets closer to 1, that is, as the quality gap in product 2 vanishes, the equilibrium prices with mixed bundling converge to those without bundling.

---

[17] See, for instance, the case of $a = 0$ or $a = 0.1$ in Table 9 in the appendix.

Proposition 8. *Assume that* $a \le b$. *We have*

$$\lim_{b \to 1} p_1^N = \lim_{b \to 1} r^N = \frac{2(1-a)}{3}, \ \lim_{b \to 1} p_2^S = 0, \text{ and } \lim_{b \to 1} r^S = \frac{1-a}{3}.$$

Proof. See Appendix. Q.E.D.

Observe that the limits are identical to the prices given in Proposition 1 for $b = 1$. When the quality gap in product 2 vanishes, it does not matter for the gross payoff whether to purchase firm N's bundle or to purchase product 1 from firm N and product 2 from firm S. This causes r^N to be equal to $p_1^N + p_2^S$. In addition, p_2^S goes down to 0 since, otherwise, firm S can increase its profits by slightly lowering the price. This implies that r^N is equal to p_1^N and that the *implicit* price of product 2 by firm N is thus zero. In short, as the two brands of one product become perfect substitutes, the firms' prices of that product go down to 0. Therefore, there is no room for bundle discounts and mixed bundling yields the same outcome as the case in Section 3 without bundling.

5 Extensions

In this section, we extend the model in the previous sections to see whether the results are robust. In particular, we relax the assumption of full market coverage and the assumption of zero marginal costs: The analysis shows that the main results continue to hold with these extensions.

5.1 The assumption of full market coverage

The analysis so far has been based on the premise that all consumers purchase one unit of each of the two products. One might wonder whether this assumption of full market coverage plays a critical role in obtaining some of the results, in particular, the results that firms' profits and/or social welfare are lower with mixed bundling. Indeed, without the full market coverage assumption, more consumers may buy the products, and so the market may expand, under mixed bundling since both the bundle prices as well as the stand-alone prices become lower. This possibility of 'market expansion' is a factor that could possibly increase firms' profits and social welfare, but that could not have been taken into account with the full market coverage assumption.

In this subsection, we relax the assumption of full market coverage and

investigate the effect of mixed bundling on prices, profits, consumer surplus and social welfare. In doing so, we assume that $v_1 = v_2 = 0$ in the payoff specification, instead of the previous assumption that both v_1 and v_2 are sufficiently high.[18] With this specification, a consumer has five more options, in addition to the four options mentioned in Section 2. Her net payoffs for these 5 options are (v) $x - p_1^N$ when she only purchases product 1 from firm N, (vi) $ax - p_1^S$ when she only purchases product 1 from firm S, (vii) $by - p_2^N$ when she only purchases product 2 from firm N, (viii) $y - p_2^S$ when she only purchases product 2 from firm S, and (ix) 0 when she purchases nothing. The net payoffs for the other 4 options are specified in Section 2. Following the notation in Section 2, denote the population of consumers who purchase only product 1 from firm N by D^{NX}, the population of consumers who purchase only product 2 from firm N by D^{XN}, and so on. Figure 4 shows the pattern of consumer choice.
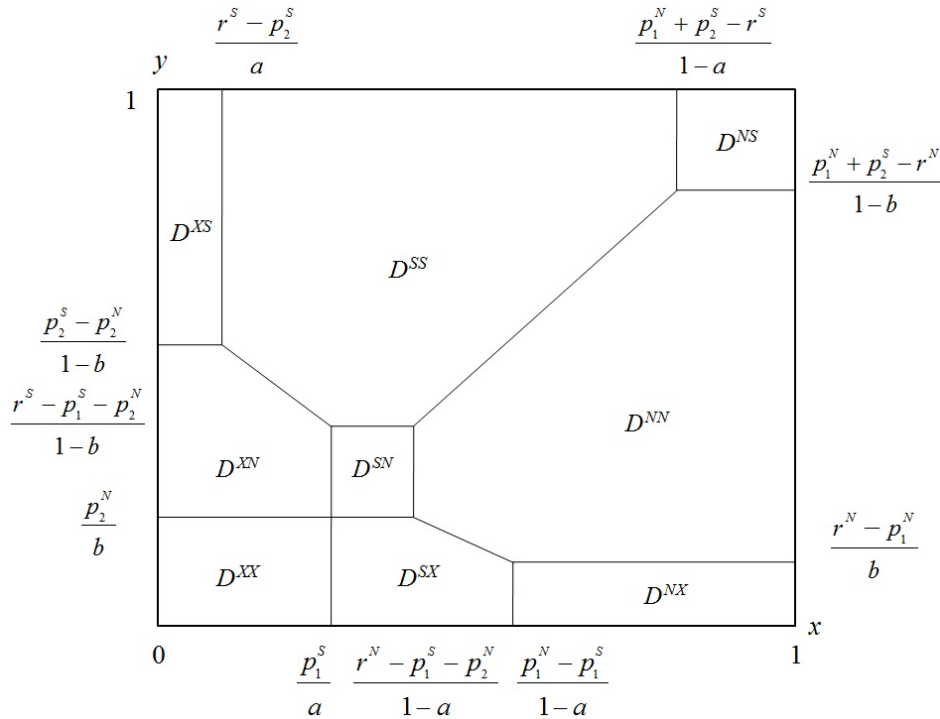

Figure 4: The pattern of consumer choice without full market coverage

---

[18] With v_1 and v_2 being nonzero but small enough, we would have qualitatively similar results. However, the equilibrium values would be slightly different because the equilibrium prices are functions of v_1 and v_2 without full market coverage.

The firms' profits are given by

$$\pi^N = p_1^N(D^{NS} + D^{NX}) + p_2^N(D^{SN} + D^{XN}) + r^N D^{NN},$$
$$\pi^S = p_1^S(D^{SN} + D^{SX}) + p_2^S(D^{NS} + D^{XS}) + r^S D^{SS}.$$

When no firm offers a bundle discount, i.e., $r^k = p_1^k + p_2^k$ for $k = N, S$, it is easy to verify from the first-order condition that the equilibrium prices, profits, consumer surplus, and social welfare are given by

$$p_1^N = \frac{2(1-a)}{4-a}, \ p_1^S = \frac{a(1-a)}{4-a}, \ p_2^N = \frac{b(1-b)}{4-b}, \ p_2^S = \frac{2(1-b)}{4-b};$$

$$\pi^N = \frac{4(1-a)}{(4-a)^2} + \frac{b(1-b)}{(4-b)^2}, \ \pi^S = \frac{a(1-a)}{(4-a)^2} + \frac{4(1-b)}{(4-b)^2};$$

$$CS = \frac{4(8+a+b)(4+a+b) + ab[5(a+b) - 88]}{2(4-a)^2(4-b)^2};$$

$$SW = \frac{4(8+a+b)(12 - 5a - 5b) + ab[15(a+b) - 4ab + 56]}{2(4-a)^2(4-b)^2}.$$

On the other hand, the equilibrium with mixed bundling is very complicated to characterize, even for the symmetric case of $a = b$. We can only provide the equilibrium outcomes numerically for given parameter values. Table 4 shows numerical values of the equilibrium outcomes when $a = b = 0.01$, $a = b = 0.5$, and $a = b = 0.99$. Note that the first row in each cell, i.e. the 'w/o' row, shows the equilibrium value without bundling, while the second row, the 'w/ ' row, shows the one with bundling.

Table 4   Equilibrium values with and without mixed bundling

		$p_1^N = p_2^S$	$p_1^S = p_2^N$	$r^N = r^S$	$\pi^N = \pi^S$	CS	SW
$a=b=0.01$	w/o	0.49624	0.00248	0.49872	0.24936	0.25440	0.75312
	w/	0.49664	0.00329	0.49667	0.24915	0.25475	0.75305
$a=b=0.5$	w/o	0.28571	0.07143	0.35714	0.18367	0.53062	0.89796
	w/	0.28125	0.08198	0.29992	0.16768	0.55942	0.89478
$a=b=0.99$	w/o	0.00664	0.00329	0.00993	0.00551	0.98785	0.99886
	w/	0.00586	0.00326	0.00672	0.00398	0.99036	0.99833

We can see that profits and social welfare are lower whereas consumer surplus is higher under mixed bundling. Therefore, the results on profits, consumer surplus, and social welfare with the assumption of full market coverage are still valid. We can also see that, as in the case of full market coverage, mixed bundling

lowers the bundle prices, r^N and r^S, and thus expands the population D^{NN} and D^{SS} of consumers who purchase both products from the same firm, but shrinks the population D^{NS} of consumers who purchase high quality brands from each firm.

However, there are some differences in the effect on the stand-alone prices. Recall that the stand-alone prices of high-quality brands are lower with mixed bundling in the case of full market coverage.[19] When the market is not fully covered, on the other hand, we obtain three different outcomes in equilibrium: (i) when the gap between the high-quality brand and the low-quality brand is large, e.g., $a = b = 0.01$, all individual prices, p_1^N, p_1^S, p_2^N and p_2^S, are higher with mixed bundling; (ii) when the quality gap is sufficiently small, e.g., $a = b = 0.99$, all individual prices lower; (iii) when the quality gap is in the intermediate range, e.g., $a = b = 0.5$, the prices of high-quality brands, p_1^N and p_2^S, are lower but the prices of low-quality brands, p_1^S and p_2^N, are higher. The reason is as follows. It is easy to see from Figures 3 and 4 that, when the market is not fully covered, firm N has an additional incentive to attract consumers in D^{NX} and D^{SX} to expand D^{NN}, and likewise for firm S. This explains why the stand-alone prices of high-quality brands might increase with mixed bundling, especially when the quality gap is large. However, if the quality gap is small and thus competition between the bundles becomes tougher, the prices of high-quality brands as well as the bundle prices have to be reduced, although the downward pressure on the former is less severe when the market is not fully covered. This explains the situations such as when $a = b = 0.5$ and $a = b = 0.99$.

We are now ready to explain why profits and social welfare are lower with mixed bundling when the market is not fully covered. First of all, mixed bundling decreases profits and social welfare by shrinking the consumer group D^{NS} who purchase high-quality brands of both products. This is because bundle prices are lower as well as the downward pressure on the stand-alone prices of high-quality brands is less severe, as explained in the previous paragraph. Secondly, it increases profits and social welfare by shrinking the consumer groups D^{NX}, D^{SX}, D^{XN}, and D^{XS} who purchase only one product. Finally, it may decrease profits and social welfare by expanding the consumer group D^{XX} who purchase neither product. As for the prices of low-quality brands, firms have two countervailing incentives. Firms have an incentive to sell more individual low-quality brands. Firms also have an incentive to keep the consumers who purchase

---

[19] The stand-alone prices of low-quality brands are irrelevant in the case of full market coverage since Proposition 2 shows that no consumer purchases low-quality brands of both products in equilibrium.

the bundles. If the latter incentive dominates the former incentive, the prices of low-quality brands would be higher, leading to an expansion of D^{XX}. This is what happens unless $a = b$ is very close to 1. All in all, as the numerical results above shows, the first effect of mixed bundling turns out to be dominant, and we get that firms' profits as well as social welfare are lower with mixed bundling.

Table 4 also suggests that the comparative static results continue to hold. That is, profits decrease while both social welfare and consumer surplus increase as $a = b$ increases. The stand-alone prices of high-quality brands as well as the bundle prices decrease as $a = b$ increases. However, the effect on the stand-alone prices of low-quality brands is rather ambiguous: p_1^S and p_2^N increase as $a = b$ increases from 0.01 to 0.5, but they decrease as $a = b$ increases from 0.5 to 0.99. This is because an increases in $a = b$ has two opposing effects. On the one hand, it enhances the quality of low-quality brands, thereby raising the demands for low-quality brands and so the stand-alone prices of low-quality brands. On the other hand, it has an effect of reducing the prices by narrowing the quality gaps and thus intensifying competition. The former effect dominates the latter when $a = b$ is small, but the situation is reversed when $a = b$ is large.

5.2 The assumption of zero marginal costs

Let us now relax the assumption of zero marginal costs. The marginal costs of producing a high-quality brand and a low-quality brand are denoted by c_H and c_L, respectively, with $c_H \geq c_L$. Then, with the assumption of full market coverage and the symmetric quality differentiation of $a = b$, we can obtain that the equilibrium outcomes without bundling are given by[20]

$$p_1^N = p_2^S = \frac{2(1-a) + 2c_H + c_L}{3}, p_1^S = p_2^N = \frac{1 - a + c_H + 2c_L}{3};$$

$$D^{NS} = \frac{[2(1-a) - (c_H - c_L)]^2}{9(1-a)^2}, D^{NN} = D^{SS} = \frac{[2(1-a) - (c_H - c_L)](1 - a + c_H - c_L)}{9(1-a)^2},$$

$$D^{SN} = \frac{(1 - a + c_H - c_L)^2}{9(1-a)^2};$$

---

[20] The quality gap must be larger than the difference in marginal costs, i.e., $c_H - c_L < 1 - a$, for the equilibrium outcomes. Otherwise, it would not be profitable to produce a high-quality brand.

24

$$\pi^N = \pi^S = \frac{5(1-a)-2(c_H-c_L)}{9} + \frac{2(c_H-c_L)^2}{9(1-a)};$$

$$SW = \frac{8+a-2(7c_H+2c_L)}{9} + \frac{5(c_H-c_L)^2}{9(1-a)}; CS = \frac{11a-2-2(5c_H+4c_L)}{9} + \frac{(c_H-c_L)^2}{9(1-a)},$$

whereas the equilibrium outcomes with mixed bundling are given by

$$p_1^N = p_2^S = \frac{7(1-a)}{12} + \frac{3c_H+c_L}{4}, \quad p_1^S = p_2^N = \frac{1-a}{3} + \frac{c_H+3c_L}{4},$$

$$r^N = r^S = \frac{2(1-a)}{3} + c_H + c_L;$$

$$D^{NS} = \frac{[1-a-(c_H-c_L)]^2}{4(1-a)^2}, \quad D^{NN} = D^{SS} = \frac{3}{8} + \frac{(c_H-c_L)(1-a+c_H-c_L)}{4(1-a)^2},$$

$$D^{SN} = \frac{(c_H-c_L)^2}{4(1-a)^2};$$

$$\pi^N = \pi^S = \frac{19(1-a)-9(c_H-c_L)}{48} + \frac{3(c_H-c_L)^2}{16(1-a)};$$

$$SW = \frac{5+a-3(c_H+c_L)}{6} + \frac{(c_H-c_L)^2}{2(1-a)}; CS = \frac{23a+1-3(9c_H+7c_L)}{24} + \frac{(c_H-c_L)^2}{8(1-a)}.$$

Observe first that, compared to the case of zero marginal costs, the population of consumers who purchase high-quality brands of both products, i.e., D^{NS}, shrinks, whereas the population of other consumer gruops, i.e., D^{NN}, D^{SS}, and D^{SN} expands. This is simply due to the fact that the marginal costs of producing high-quality brands and so its prices are higher. Observe next that Proposition 6 continues to hold. That is, (i) prices, profits and social welfare are lower, whereas (ii) consumer surplus as well as the population of consumers who purchase both products from the same firm is larger in the equilibrium with mixed bundling. It is a straightforward exercise to observe that the comparative static results of the previous sections do not change, either.[21]

---

[21] We conjecture that the assumption of zero marginal costs does not have a significant bite even for the case when quality differentiation is asymmetric and/or the case when the market is not fully covered.

6 Conclusion

We have analyzed the effects of competitive mixed bundling in vertically differentiated product markets. We have shown that, compared to the equilibrium without bundling, (i) prices, profits and social welfare are lower, whereas (ii) consumer surplus is higher in the equilibrium with mixed bundling. In addition, the population of consumers who purchase both products from the same firm is larger in the equilibrium with mixed bundling. These results are largely in line with those obtained in the previous literature.

Further, when the quality gap between brands narrows under no bundling and symmetric mixed bundling, prices and profits decrease. The reason is basically that firms compete more intensely when brands become more substitutable. We also find that the magnitude of reduction in profits as a result of mixed bundling decreases when the quality gap increases. In other words, the profit-reducing effect of mixed bundling gets exacerbated as the degree of vertical differentiation becomes higher.

Asymmetric quality differentiation has yielded more interesting observations with regard to the comparative statics. As shown in Section 4.2, an increase in a reduces both firms' profits, whereas an increase in b increases firm N's profits but reduces firm S's profits. This asymmetry is due to the difference in the effect of changes in quality parameters on the firms' competitive advantage: An increase in a narrows the quality gap between the firms' bundles, whereas an increase in b widens it. It has also been shown that prices react to changes in these quality parameters in an interesting way. We have finally shown that the equilibrium prices with mixed bundling converge to those without bundling when the quality gap in one product vanishes, that is, when b gets close to one.

We want to finish by pointing out some limitations of the present paper and directions for further research. We assumed that consumers' valuations for the two products are independent. With this *independent valuations* assumption, we have shown that competition between the firms is completely separated across the products under no bundling, whereas strategic interdependences between the products are created under mixed bundling. This is the reason why profits decrease under mixed bundling. However, if consumers' valuations for the two products are correlated, strategic interdependences between the products exist even without bundling. Then the role of mixed bundling in intensifying competition would be diminished, and thus, the effects of mixed bundling on profits, social welfare, and consumer surplus might be ambiguous. It is an interesting topic to examine mixed bundling under general distributions of consumers' valuations.

Next, the present paper considered the case in which one firm offers a high-quality brand of one product and the other offers a high-quality brand of another

product. This is the case of *shared quality leadership*, according to the terminology of Einhorn (1992). It is also likely in reality that one firm, the quality leader, offers high-quality brands of both products and the other, the quality follower, offers low-quality brands of both products. In this alternative case of *complete quality leadership*, again according to Einhorn's terminology, the aspects of competition would be different because standpoints of firms change. In the case of shared quality leadership, our result shows that consumers with high valuations for both products purchase high-quality brands of both firms, whereas consumers with a high valuation for one product but a low valuation for the other product purchase a bundle. Thus, the mixed bundling outcome in which some consumers purchase bundles and others purchase the products from both firms occurs in equilibrium.[22] In the case of complete quality leadership, however, it is not clear whether there exists a 'mixed bundling equilibrium'. The intuition is as follows. In a mixed bundling equilibrium, if it exists, a natural configuration would be that consumers with a high valuation for one product but a low valuation for the other product purchase the former product from the quality leader and the latter from the quality follower, while consumers with high (low, respectively) valuations for both products purchase a bundle of the quality leader (the quality follower, respectively). However, note that the quality leader may wish to induce all consumers to purchase one of the firms' bundles by offering deeper bundle discounts, i.e. by reducing the bundle price. In this regime of 'pure bundling,' the aspect of competition changes to a competition between the high-quality bundle and the low-quality bundle: consumers with higher *total* valuations for the two products purchase the bundle of the quality leader while those with lower *total* valuations purchase the bundle of the quality follower. Although competition becomes tougher, the quality leader might prefer the pure bundling regime since it could attract the consumers who have higher *total* valuations for the products. On the other hand, the quality follower might wish to avoid this type of competition since it would only attract the consumers who have lower *total* valuations for the products. This tension between two firms seems to be important in characterizing the equilibrium in the case of complete quality leadership. However, it is not an easy task to delve into this issue within the present framework. We leave a fuller analysis of this case to future research.

Appendix

This appendix contains all the omitted proofs as well as tables for numerical results. The proofs are given in the order of proposition, except that the proof of

---

[22] See Proposition 3 and the preceding discussion.

Proposition 4 appears after that of Proposition 5 and a technical lemma given below since these are needed for proving Proposition 4.

Proof of Proposition 2

(i) Consider first the case when $\min\{r^N, r^S\} = r^S$, that is, $r^S \le r^N$. Suppose to the effect of contradiction that $r^S > p_1^S + p_2^N$. We will show that either firm N can increase its profits by raising p_2^N to $p_2^{N'} = r^S - p_1^S$ or firm S can increase its profits by raising p_1^S to $p_1^{S'} = r^S - p_2^N$. From Figure 2, the firms' profits are

$$
\begin{aligned}
\pi^N &= p_1^N D^{NS} + p_2^N D^{SN} + r^N D^{NN} \\
&= p_1^N (1 - \frac{p_1^N + p_2^S - r^S}{1-a})(1 - \frac{p_1^N + p_2^S - r^N}{1-b}) + p_2^N \frac{r^N - p_1^S - p_2^N}{1-a} \frac{r^S - p_1^S - p_2^N}{1-b} \\
&\quad + r^N \{ \frac{p_1^N + p_2^S - r^N}{1-b}(1 - \frac{p_1^N + p_2^S - r^S}{1-a}) \\
&\quad + \frac{1}{2}(\frac{p_1^N + p_2^S - r^S}{1-a} - \frac{r^N - p_1^S - p_2^N}{1-a})(\frac{p_1^N + p_2^S - r^N}{1-b} + \frac{r^S - p_1^S - p_2^N}{1-b}) \},
\end{aligned}
$$

$$
\begin{aligned}
\pi^S &= p_1^S D^{SN} + p_2^S D^{NS} + r^S D^{SS} \\
&= p_1^S \frac{r^N - p_1^S - p_2^N}{1-a} \frac{r^S - p_1^S - p_2^N}{1-b} + p_2^S (1 - \frac{p_1^N + p_2^S - r^S}{1-a})(1 - \frac{p_1^N + p_2^S - r^N}{1-b}) \\
&\quad + r^S \{ \frac{p_1^N + p_2^S - r^S}{1-a}(1 - \frac{p_1^N + p_2^S - r^N}{1-b}) \\
&\quad + \frac{1}{2}(\frac{p_1^N + p_2^S - r^N}{1-b} - \frac{r^S - p_1^S - p_2^N}{1-b})(\frac{p_1^N + p_2^S - r^S}{1-a} + \frac{r^N - p_1^S - p_2^N}{1-a}) \}.
\end{aligned}
$$

Now consider the situation in which firm S raises p_1^S to $p_1^{S'} = r^S - p_2^N$ while keeping p_2^S and r^S fixed. Then Figure 3(a) becomes relevant, and firm S's new profits are

$$\pi^{S'} = p_2^S D^{NS} + r^S D^{SS} = p_2^S\left(1 - \frac{p_1^N + p_2^S - r^S}{1-a}\right)\left(1 - \frac{p_1^N + p_2^S - r^N}{1-b}\right)$$

$$+ r^S\left\{\frac{p_1^N + p_2^S - r^S}{1-a}\left(1 - \frac{p_1^N + p_2^S - r^N}{1-b}\right)\right.$$

$$\left.+ \frac{1}{2}\frac{p_1^N + p_2^S - r^N}{1-b}\left(\frac{p_1^N + p_2^S - r^S}{1-a} + \frac{r^N - r^S}{1-a}\right)\right\}.$$

Thus, we have

$$\pi^{S'} - \pi^S = \frac{r^S - p_1^S - p_2^N}{2(1-a)(1-b)}[2(r^S - p_1^S)(r^N - r^S) + (r^S - 2p_1^S)(r^S - p_1^S - p_2^N)]$$

$$= \frac{r^S - p_1^S - p_2^N}{2(1-a)(1-b)}[p_2^N(2p_1^S - r^S) - (r^S - p_1^S)(2p_1^S + r^S - 2r^N)]. \tag{1}$$

We divide the cases below. First, if $r^S > 2p_1^S$ then we can easily see from the first equality that $\pi^{S'} > \pi^S$ since $r^S \le r^N$ and $r^S > p_1^S + p_2^N$ by supposition. Second, if $r^S < 2p_1^S$ and $2p_1^S + r^S \le 2r^N$ then we can also easily see from the second equality that $\pi^{S'} > \pi^S$ since $p_2^N > 0$.[23] Third, consider the case when $r^S < 2p_1^S$ and $2p_1^S + r^S > 2r^N$. Since the sign of $\pi^{S'} - \pi^S$ is the same as that of

$$p_2^N - \frac{(r^S - p_1^S)(2p_1^S + r^S - 2r^N)}{2p_1^S - r^S},$$

we would have $\pi^{S'} \le \pi^S$ when $p_2^N \le (r^S - p_1^S)(2p_1^S + r^S - 2r^N)/(2p_1^S - r^S)$. If the last inequality holds, however, it can be established that firm N can increase its profits by raising p_2^N to $p_2^{N'} = r^S - p_1^S$ while keeping p_1^N and r^N fixed. By doing so, firm N's profits change to

---

[23] If $p_2^N = 0$, then $\partial\pi^N/\partial p_2^N = [3(p_2^N)^2 - (3r^N + 2r^S - 4p_1^S)p_2^N + (r^S - p_1^S)(2r^N - p_1^S)]/(1-a)$ $(1-b) > 0$ under our supposition of $p_1^S + p_2^N < r^S \le r^N$. Thus, $p_2^N = 0$ cannot be a part of the equilibrium.

$$\pi^{N\prime} = p_1^N D^{NS} + r^N D^{NN} = p_1^N (1 - \frac{p_1^N + p_2^S - r^S}{1-a})(1 - \frac{p_1^N + p_2^S - r^N}{1-b})$$

$$+ r^N \left\{ \frac{1}{2} \frac{p_1^N + p_2^S - r^N}{1-b} (1 - \frac{p_1^N + p_2^S - r^S}{1-a} + 1 - \frac{r^N - r^S}{1-a}) \right\},$$

and we get

$$\pi^{N\prime} - \pi^N = \frac{r^S - p_1^S - p_2^N}{2(1-a)(1-b)} [2(p_2^N)^2 - (3r^N - 2p_1^S)p_2^N + (r^S - p_1^S)r^N]. \tag{2}$$

Since $r^S > p_1^S + p_2^N$ by supposition, the sign of $\pi^{N\prime} - \pi^N$ is the same as that of $2(p_2^N)^2 - (3r^N - 2p_1^S)p_2^N + (r^S - p_1^S)r^N$, and it is an easy exercise to show that this quadratic form is positive when $p_2^N \leq (r^S - p_1^S)(2p_1^S + r^S - 2r^N)/(2p_1^S - r^S)$. Fourth, consider the case when $r^S = 2p_1^S$. Note that

$$\pi^{S\prime} - \pi^S = \frac{r^S(r^S - p_1^S - p_2^N)(r^N - r^S)}{2(1-a)(1-b)}$$

is strictly positive when $r^N > r^S$, and we are done. If, on the other hand, $r^N = r^S$ holds then we have $\pi^{S\prime} = \pi^S$. In that case, that is, when $r^S = 2p_1^S$ and $r^N = r^S$ hold, we have

$$\pi^{N\prime} - \pi^N = \frac{(r^S - p_1^S - p_2^N)(2p_2^N - r^N)^2}{2(1-a)(1-b)},$$

which is positive unless $r^N = 2p_2^N$ and we are done. Finally, observe that the last possibility of having $r^S = 2p_1^S$, $r^N = 2p_2^N$, and $r^N = r^S$ contradicts to our supposition of $r^S > p_1^S + p_2^N$. This completes the proof for the case of $r^S \leq r^N$.

(ii) The proof for the case when $\min\{r^N, r^S\} = r^N$ is symmetric to the previous one. Suppose to the effect of contradiction that $r^N > p_1^S + p_2^N$. Consider the situation in which firm N raises p_2^N to $p_2^{N\prime\prime} = r^N - p_1^S$ while keeping p_1^N and r^N fixed. Then Figure 3(b) becomes relevant, and firm N's new profits are

$$\pi^{N''} = p_1^N D^{NS} + r^N D^{NN} = p_1^N \left(1 - \frac{p_1^N + p_2^S - r^S}{1-a} \right) \left(1 - \frac{p_1^N + p_2^S - r^N}{1-b} \right)$$
$$+ r^N \left\{ \frac{p_1^N + p_2^S - r^N}{1-b} \left(1 - \frac{p_1^N + p_2^S - r^S}{1-a} \right) \right.$$
$$\left. + \frac{1}{2} \frac{p_1^N + p_2^S - r^S}{1-a} \left(\frac{p_1^N + p_2^S - r^N}{1-b} + \frac{r^S - r^N}{1-b} \right) \right\}.$$

Thus, we have

$$\pi^{N''} - \pi^N = \frac{r^N - p_1^S - p_2^N}{2(1-a)(1-b)} [2(r^N - p_2^N)(r^S - r^N) + (r^N - 2p_2^N)(r^N - p_1^S - p_2^N)]$$
$$= \frac{r^N - p_1^S - p_2^N}{2(1-a)(1-b)} [p_1^S(2p_2^N - r^N) - (r^N - p_2^N)(2p_2^N + r^N - 2r^S)]. \tag{3}$$

When firm S raises p_1^S to $p_1^{S''} = r^N - p_2^N$ while keeping p_2^S and r^S fixed, its profits change to

$$\pi^{S''} = p_2^S D^{NS} + r^S D^{SS} = p_2^S (1 - \frac{p_1^N + p_2^S - r^S}{1-a})(1 - \frac{p_1^N + p_2^S - r^N}{1-b})$$
$$+ r^S \left\{ \frac{1}{2} \frac{p_1^N + p_2^S - r^S}{1-a} (1 - \frac{p_1^N + p_2^S - r^N}{1-b} + 1 - \frac{r^S - r^N}{1-b}) \right\},$$

and we get

$$\pi^{S''} - \pi^S = \frac{r^N - p_1^S - p_2^N}{2(1-a)(1-b)} [2p_1^S - (3r^S - 2p_2^N)p_1^S + (r^N - p_2^N)r^S]. \tag{4}$$

Observe that (3) and (4) are symmetric to (1) and (2) when r^N, r^S, p_1^S, and p_2^N are replaced by r^S, r^N, p_2^N, and p_1^S, respectively. Thus, the same argument as the previous one applies to show that either $\pi^{N''} - \pi^N > 0$ or $\pi^{S''} - \pi^S > 0$ holds. This completes the proof.   Q.E.D.

Proof of Proposition 3

Suppose either $(p_1^N + p_2^S - r^S)/(1-a) \geq 1$ or $(p_1^N + p_2^S - r^N)/(1-b) \geq 1$ holds.

Assume without loss of generality that $a \leq b$. Assume further that $r^S \leq r^N$. We can in fact show that $r^S > r^N$ cannot hold when $a \leq b$ in any equilibrium with pure bundling.[24] Consumer choice can be depicted by a diagram similar to Figure 3, with two cases that depend on whether $r^N - r^S \leq b - a$ or $r^N - r^S > b - a$. When $r^N - r^S > b - a$, the firms' profits are

$$\pi^N = r^N D^{NN} = r^N \left\{ \frac{1}{2} \frac{1 - a + r^S - r^N}{1 - b} (1 - \frac{r^N - r^S}{1 - a}) \right\},$$

$$\pi^S = r^S D^{SS} = r^S \left\{ 1 - \left(\frac{1}{2} \frac{1 - a + r^S - r^N}{1 - b} (1 - \frac{r^N - r^S}{1 - a}) \right) \right\}.$$

It is straightforward to show that we cannot have $r^N - r^S > b - a$ at r^N and r^S satisfying the first-order condition. Thus, we must have $r^N - r^S \leq b - a$ and the firms' profits in this case are

$$\pi^N = r^N D^{NN} = r^N \left\{ \frac{1}{2} (1 - \frac{1 - b + r^N - r^S}{1 - a} + 1 - \frac{r^N - r^S}{1 - a}) \right\},$$

$$\pi^S = r^S D^{SS} = r^S \left\{ \frac{1}{2} (\frac{1 - b + r^N - r^S}{1 - a} + \frac{r^N - r^S}{1 - a}) \right\}.$$

From the first-order condition, we get the equilibrium outcome of $r^N = (3 - 4a + b)/6$, $r^S = (3 - 2a - b)/6$, $\pi^N = (3 - 4a + b)^2/36(1 - a)$, and $\pi^S = (3 - 2a - b)^2/36(1 - a)$. Then the inequality $\max \left\{ (p_1^N + p_2^S - r^S)/(1 - a), (p_1^N + p_2^S - r^N)/(1 - b) \right\} \geq 1$ becomes

$$p_1^N + p_2^S \geq \min \left\{ \frac{9 - 4a - 5b}{6}, \frac{9 - 8a - b}{6} \right\} = \frac{9 - 4a - 5b}{6}.$$

Now suppose that firm S lowers p_2^S to $p_2^{S'} = (9 - 4a - 5b)/6 - p_1^N - \varepsilon$, where ε is a small positive real number, while keeping $r^S = (3 - 2a - b)/6$ fixed. Then Figure 3(a) becomes relevant, and the difference between firm S's new profit, denoted by $\pi^{S'}$, and the current one is

---

[24] The proof is straightforward and thus omitted. We show in the proof of Proposition 4 that $r^S \leq r^N$ holds when $a \leq b$ in any equilibrium with mixed bundling.

$$\pi^{S\prime} - \pi^S = \frac{\varepsilon[-12(2(b-a)+3\varepsilon)p_1^N - 36\varepsilon^2 + \varepsilon(45-51b+6a) + 4(b-a)(9-4a-5b)]}{36(1-a)(1-b)}.$$

Since $p_1^N \le r^N = (3-4a+b)/6$,[25] we have

$$-12(2(b-a)+3\varepsilon)p_1^N - 36\varepsilon^2 + \varepsilon(45-51b+6a) + 4(b-a)(9-4a-5b)$$
$$\ge -12(2(b-a)+3\varepsilon)r^N - 36\varepsilon^2 + \varepsilon(45-51b+6a) + 4(b-a)(9-4a-5b)$$
$$= 24(b-a)(1-b) + (27-57b+30a)\varepsilon - 36\varepsilon^2.$$

When $a < b$, the last expression is positive for ε sufficiently close to 0. When $a = b$, it becomes $27(1-a)\varepsilon - 36\varepsilon^2$, which is also positive for ε sufficiently close to 0. Hence, firm S can profitably deviate to $p_2^{S\prime}$ by taking sufficiently small ε. Therefore, there is no equilibrium in which $(p_1^N + p_2^S - r^S)/(1-a) \ge 1$ or $(p_1^N + p_2^S - r^N)/(1-b) \ge 1$ holds.   Q.E.D.

Proof of Proposition 5

The first-order condition is given by the following four equations.

$$\frac{\partial \pi^N}{\partial p_1^N} = \frac{1}{(1-a)(1-b)}[3(p_1^N)^2 - (3r^N + 2r^S - 4p_2^S + 4 - 2a - 2b)p_1^N$$
$$+ r^N(2r^S - 2p_2^S + 2 - 2a) + (r^S - p_2^S + 1 - a)(1 - b - p_2^S)] = 0, \tag{5}$$

$$\frac{\partial \pi^N}{\partial r^N} = \frac{1}{2(1-a)(1-b)}[3(r^N)^2 - 4(1-a+r^S)r^N + 2r^S(2p_1^N + p_2^S) - 3(p_1^N)^2$$
$$+ 4p_1^N(1-a-p_2^S) + p_2^S(2 - 2a - p_2^S)] = 0, \tag{6}$$

$$\frac{\partial \pi^S}{\partial p_2^S} = \frac{1}{(1-a)(1-b)}[3(p_2^S)^2 - (3r^S + 2r^N - 4p_1^N + 4 - 2a - 2b)p_2^S$$
$$+ r^S(2r^N - 2p_1^N + 2 - 2b) + (r^N - p_1^N + 1 - b)(1 - a - p_1^N)] = 0, \text{ and} \tag{7}$$

$$\frac{\partial \pi^S}{\partial r^S} = \frac{1}{2(1-a)(1-b)}[-4(1-b)r^S - (r^N)^2 + 2r^N(2p_2^S + p_1^N) - 3(p_2^S)^2$$
$$+ 4p_2^S(1 - b - p_1^N) + p_1^N(2 - 2b - p_1^N)] = 0. \tag{8}$$

_____

[25] If $p_1^N > r^N$, then the price p_2^N of product 2 becomes negative. Recall that we ruled out negative prices.

When $a = b$, it is straightforward to see that $p_1^N = p_2^S$ and $r^N = r^S$ by inspecting (5)-(8). Let $p = p_1^N = p_2^S$ and $r = r^N = r^S$. Then, from (5) and (6), we get

$$2(1-a)(3p-2r)-(4p-r)(2p-r) = 0 \text{ and}$$
$$(2r-4p+1-a)(r-2p+1-a) = 0.$$

The solutions for these equations are

$$p = \frac{3(1-a)}{4}, \quad r = \frac{1-a}{2}; \text{ and } p = \frac{7(1-a)}{12}, \quad r = \frac{2(1-a)}{3}.$$

The first solution does not satisfy the second-order sufficient condition. To see this, observe from (5) that

$$\frac{\partial \pi^N}{\partial p_1^N} = \frac{[12p_1^N - 5(1-a)][4p_1^N - 3(1-a)]}{16(1-a)^2} \text{ and}$$

$$\frac{\partial^2 \pi^N}{\partial (p_1^N)^2} = \frac{12p_1^N - 7(1-a)}{2(1-a)^2}$$

when $p_2^S = 3(1-a)/4$ and $r^N = r^S = (1-a)/2$. Thus, $p_1^N = 3(1-a)/4$ does not satisfy the second-order condition. It is easy to see that the remaining solution of $p = 7(1-a)/12$ and $r = 2(1-a)/3$ satisfies the second-order sufficient condition. It is also easy to derive the rest of the equilibrium outcome. Q.E.D.

Characterization of equilibrium when $a \le b$

For notational simplicity, let

$$A = 1-a, \quad B = 1-b, \text{ and } K = B[16(16A+9B)(p_2^S)^2 - 312ABp_2^S + 121A^2B].$$

Thus, A is the quality gap in product 1 and B is the quality gap in product 2. We first establish a technical lemma.

Lemma . *Assume that $a \le b$. The equilibrium prices are given as follows:*
(i) p_2^S is the smaller of the two positive real solutions for the cubic equation

$$f(p_2^S) = 576(A - B)(p_2^S)^3 - 24B(37A - 9B)(p_2^S)^2 - 2AB(605A - 387B)p_2^S$$
$$+ 3AB^2(242A - 81B) = 0,$$

(ii) $p_1^N = \dfrac{1}{4B(16p_2^S - 3B)}[(2p_2^S - B)\sqrt{K} - B\{24(p_2^S)^2 - 6(9A + 2B)p_2^S + 17AB\}]$,

(iii) $r^N = \dfrac{1}{32Bp_2^S(16p_2^S - 3B)}[(4p_2^S - B)(4p_2^S + 3B)\sqrt{K} + B\{320(p_2^S)^3$
$$+ 48(9A - 4B)(p_2^S)^2 - 12B(26A - 3B)p_2^S + 33AB^2\}], \text{ and}$$

(iv) $r^S = \dfrac{1}{8B(16p_2^S - 3B)}[(4p_2^S - B)\sqrt{K} + B\{80(p_2^S)^2 - 12(A + B)p_2^S - 5AB\}]$.

Some remarks on the lemma are in order. First of all, observe that $f(p_2^S) = 0$ given in part (i) of the lemma becomes $-A^2[12p_2^S - 7A][56p_2^S + 69A] = 0$ when $a = b$. Thus we get $p_2^S = 7A/12$ and, after putting this into parts (ii), (iii), and (iv) of the lemma, we subsequently get $p_1^N = 7A/12$ and $r^N = r^S = 2A/3$. This is the solution obtained in Proposition 5 for the symmetric mixed bundling case. Secondly, when $a < b$, the equation $f(p_2^S) = 0$ has two positive and one negative real solutions. Note that $576(A - B)$, the coefficient of $(p_2^S)^3$, and the constant term $3AB^2(242A - 81B)$ are positive. Moreover, it is straightforward to see that $f(7B/12) \geq 0$, $f'(7B/12) < 0$, and $f(2B/3) < 0$.[26] Thus, we conclude that the equation $f(p_2^S) = 0$ has two positive real solutions, with the smaller one lying in the half-open interval $[7B/12, 2B/3)$, and one negative solution. Thirdly, the following proof shows that the larger of the two positive real solutions for $f(p_2^S) = 0$ violates the second-order condition, whereas the smaller one satisfies it. Hence, the equilibrium price of p_2^S is the smaller of the two positive real solutions for $f(p_2^S) = 0$, as specified in part (i) of the lemma.

Proof of Lemma

The first-order condition is given by equations (5) - (8) in the proof of Proposition 5. From (8), r^S can be rewritten as

---

[26] From part (i) of the lemma, we have $f(7B/12) = B^2(121A + 245B)(A - B)/6 \geq 0$, $f'(7B/12) = -2B(605A^2 - 163AB + 168B^2) < 0$, and $f(2B/3) = -B^2(242A^2 - 147AB + 224B^2)/3 < 0$.

$$r^S = \frac{2B(p_1^N + 2p_2^S) - (p_1^N + p_2^S - r^N)(p_1^N + 3p_2^S - r^N)}{4B}.$$

Put this back into (5), (6) and (7), and denote the resulting equations by (5)', (6)' and (7)', respectively. Add (5)' and (6)' to get

$$5(r^N)^2 - 2(5p_1^N + 2p_2^S)r^N + (5p_1^N - p_2^S)(p_1^N + p_2^S) + 4AB - 6p_1^N = 0. \tag{9}$$

Add (6)' and (7)' to get

$$\begin{aligned}
&(2B + p_2^S)(r^N)^2 - 2(AB + p_1^N p_2^S + 2p_2^S)r^N + 2AB(p_1^N - p_2^S) \\
&+ 2B((p_1^N)^2 + (p_2^S)^2 + p_1^N p_2^S) + p_2^S(p_1^N + 3p_2^S)(p_1^N + p_2^S) + 2AB^2 = 0.
\end{aligned} \tag{10}$$

Now solve for r^N by using (9) and (10) to get

$$r^N = \frac{10B(p_1^N)^2 - B(5A + 6B)p_1^N - 8(p_2^S)^3 + 4B(p_2^S)^2 + 7ABp_2^S - AB^2 + 2p_1^N p_2^S(3B - 4p_2^S)}{10Bp_1^N + 4Bp_2^S - 8(p_2^S)^2 - 5AB}. \tag{11}$$

Subtract (7) from (5) to get

$$r^S = \frac{(2p_1^N - B)r^N + (A + B)(p_1^N - p_2^S) - 2(p_1^N + p_2^S)(p_1^N - p_2^S)}{2p_2^S - B}.$$

Insert (11) into this to get

$$r^S = \frac{2B(p_1^N)^2 + ABp_1^N - 8(p_2^S)^3 + 4(2A + B)(p_2^S)^2 - 5ABp_2^S - A^2B + 2p_1^N p_2^S(3B - 4p_2^S)}{10Bp_1^N + 4Bp_2^S - 8(p_2^S)^2 - 5AB}. \tag{12}$$

Observe that (11) and (12) solve for r^N and r^S as functions of p_1^N and p_2^S. Now insert (11) and (12) back into (5) to get

$$\frac{2[2p_1^N + 2p_2^S - 3A]B}{AB[10Bp_1^N + 4Bp_2^S - 8(p_2^S)^2 - 5AB]^2} \times g(p_1^N, p_2^S) = 0,$$

where

$$g(p_1^N, p_2^S) = 2B(16p_2^S - 3B)(p_1^N)^2 + B[24(p_2^S)^2 - 6(9A + 2B)p_2^S + 17AB]p_1^N$$
$$- 8A(p_2^S)^3 - 4AB(p_2^S)^2 + AB(19A + 4B)p_2^S - 7A^2B^2.$$

Thus, we have either $2(p_1^N + p_2^S) = 3A$ or $g(p_1^N, p_2^S) = 0$. Suppose first that $2(p_1^N + p_2^S) = 3A$ holds in equilibrium. Putting this relation into (11) and (12) gives $r^N = (3A - 2B)/2$ and $r^S = A/2$. However, since it is straightforward to see that

$$p_2^S = \frac{3A}{2} - p_1^N, \quad r^N = \frac{3A - 2B}{2}, \text{ and } r^S = \frac{A}{2}$$

are not consistent with the second-order condition of $\partial^2 \pi^N / \partial(p_1^N)^2 \le 0$ and $\partial^2 \pi^S / \partial(p_2^S)^2 \le 0$, we must have $g(p_1^N, p_2^S) = 0$ in equilibrium. Observe that this equation is written as a quadratic equation of p_1^N and so has two solutions, say $\underline{p}_1^N$ and $\overline{p}_1^N$. We will show at the end of the proof that the smaller solution $\underline{p}_1^N$ cannot satisfy the inequality given in Proposition 3. Thus, we obtain p_1^N in part (ii) of the lemma as the larger solution $\overline{p}_1^N$ of $g(p_1^N, p_2^S) = 0$. Insert this p_1^N into (11) and (12), and we get r^N and r^S as in parts (iii) and (iv) of the lemma.

To derive the cubic equation $f(p_2^S) = 0$ given in part (i) of the lemma, insert parts (ii), (iii) and (iv) into (8). Then, we get

$$\frac{\partial \pi^S}{\partial r^S} = \frac{Q\sqrt{K} + R}{1024A(p_2^S)^2(16p_2^S - 3B)} = 0, \tag{13}$$

where

$$Q = 96(p_2^S)^2 + 4(44A - 9B)p_2^S - 33AB,$$
$$R = 384(8A - 3B)(p_2^S)^3 - 48B(26A - 9B)(p_2^S)^2 - 8AB(242A + 9B)p_2^S + 363A^2B^2,$$
and
$$K = B[16(16A + 9B)(p_2^S)^2 - 312ABp_2^S + 121A^2B].$$

Thus, we must have $Q\sqrt{K} + R = 0$ in equilibrium. We will shortly show that $Q > 0$ and $R < 0$ in equilibrium. This implies that

$$Q^2K - R^2 = -1024A(p_2^S)^2(16p_2^S - 3B) \times f(p_2^S) = 0,$$

where

$$f(p_2^S) = 576(A-B)(p_2^S)^3 - 24B(37A-9B)(p_2^S)^2$$
$$- 2AB(605A-387B)p_2^S + 3AB^2(242A-81B).$$

We cannot have $p_2^S = 0$ or $p_2^S = 3B/16$ in equilibrium. If $p_2^S = 0$, then p_1^N, r^N, and r^S given in parts (ii), (iii) and (iv) of the lemma become $7A/3$, $(56A-9B)/24$, and $2A/3$, respectively. Then, it is easy to show that they are not consistent with the first-order condition. If $p_2^S = 3B/16$, then the numerator of p_1^N in part (ii) of the lemma has a negative value, whereas the denominator of p_1^N becomes zero. Therefore, we have $f(p_2^S) = 0$ in equilibrium.

As discussed just before the proof, the equation $f(p_2^S) = 0$ has two positive real solutions, with the smaller one lying in the half-open interval $[7B/12, 2B/3)$, and one negative solution. We now show that the larger of the two positive real solutions for $f(p_2^S) = 0$ violates the second-order condition. In particular, we will show that $\partial^2 \pi^N / \partial(r^N)^2 > 0$ at the larger solution. From equation (6) in the proof of Proposition 5, i.e., from the expression of $\partial \pi^N / \partial r^N$, we have

$$\frac{\partial^2 \pi^N}{\partial(r^N)^2} = -\frac{2A - 3r^N - 2r^S}{AB} = \frac{X + Y\sqrt{K}}{16AB^2(16p_2^S - 3B)p_2^S},$$

where

$$X = B[320(p_2^S)^3 + 16(11A-30B)(p_2^S)^2 - 4(176A-27B)Bp_2^S + 99AB^2],$$
$$Y = (4p_1^S + 9B)(4p_1^S - B), \text{ and}$$
$$K = B[16(16A+9B)(p_2^S)^2 - 312ABp_2^S + 121A^2B].$$

The second equality obtains by inserting parts (iii) and (iv) of the lemma into r^N and r^S. Since the smaller of the two positive real solutions for $f(p_2^S) = 0$ is greater than $7B/12$, both the denominator and Y are positive. We will show that X is positive at the larger solution (whereas it is negative at the smaller solution). Observe that X is written as a cubic form of p_2^S. Now, by part (i) of the lemma, $(p_2^S)^3$ can be written as

$$(p_2^S)^3 = \frac{24B(37A-9B)(p_2^S)^2 + 2AB(605A-387B)p_2^S - 3AB^2(242A-81B)}{576(A-B)}.$$

Further, if we let $B = kA$, where $0 \leq k < 1$, then X becomes $(kA^2/(1-k))h(p_2^S)$, where

$$h(p_2^S) = 24(66 - 61k + 135k^2)(p_2^S)^2 - 2(143 - 1719k + 486k^2)kAp_2^S$$
$$- 3kA^2(913 - 108k).$$

Hence, the sign of X is equal to that of $h(p_2^S)$. Since the sign of the coefficient of $(p_2^S)^2$ in $h(p_2^S)$ is positive but the constant term, $-3kA^2(913 - 108k)$, is negative, the quadratic equation $h(p) = 0$ has one negative and one positive real solution. Denote the positive solution by $\hat{p}$.[27] If $\hat{p}$ lies between two positive solutions of $f(p_2^S) = 0$, the sign of X is positive at the larger solution and is negative at the smaller one. To show this, note that it suffices to establish that $f(\hat{p}) < 0$, given the shape of $f(\cdot)$. Observe that

$$f(\hat{p}) = -\frac{k^2 A^4}{24(66 - 61k + 135k^2)^3}[M + N\sqrt{L}],$$

where

$$M = 28697814k^7 - 2117792385k^6 + 3543156513k^5 + 3619710261k^4$$
$$+ 3484409643k^3 - 2157062886k^2 + 2299124146k - 1063902906,$$
$$N = 59049k^5 + 3363606k^4 - 7156107k^3 + 5607558k^2 - 2891416k + 1317690, \text{ and}$$
$$L = 236196k^4 - 2720628k^3 + 12442653k^2 - 5014746k + 4359025.$$

It can be checked that we have $N > 0$ for all $k \in [0,1)$, while $M > 0$ for $k \in (0.4386, 1)$ and $M < 0$ for $k \in [0, 0.4386)$. Further, when $k \in [0, 0.4386)$, we have $N^2L - M^2 > 0$. This establishes that $M + N\sqrt{L} > 0$ and so $f(\hat{p}) < 0$. Proceeding similarly, we can in fact check that the smaller solution satisfies the second-order condition.

Now let us confirm that $Q > 0$ and $R < 0$ in equilibrium. Observe first that

---

[27] $\hat{p} = kA(143 - 1719k + 486k^2 + \sqrt{L})/24(66 - 61k + 135k^2))$, where L is defined below.

the value of Q at $p_2^S = 7B/12$ is $B(209A+35B)/3 > 0$. It is then easy to see that $Q > 0$ for all $p_2^S \in [7B/12, 2B/3)$ since Q is strictly increasing in p_2^S. Observe next that R is a cubic form of p_2^S, where the coefficient of $(p_2^S)^3$ as well as the constant term is positive. Thus, R has at most two positive real solutions. Since the values of R evaluated at $p_2^S = 7B/12$ and at $p_2^S = 2B/3$ are both negative, we have $R < 0$ for all $p_2^S \in [7B/12, 2B/3)$.[28]

Finally, we show that the smaller solution $\underline{p}_1^N$ of $g(p_1^N, p_2^S) = 0$ does not satisfy the inequality in Proposition 3. In particular, we will show that $\underline{p}_1^N + p_2^S - r^N > B$ holds. Insert $\underline{p}_1^N$ into (11) and (12) to get another set of values, $\underline{r}^N$ and $\underline{r}^S$, as follows:

$$\underline{p}_1^N = \frac{1}{4B(16p_2^S - 3B)}[-(2p_2^S - B)\sqrt{K} - B(24(p_2^S)^2 - 6(9A+2B)p_2^S + 17AB)],$$

$$\underline{r}^N = \frac{1}{32Bp_2^S(16p_2^S - 3B)}[-(4p_2^S - B)(4p_2^S + 3B)\sqrt{K} + B(320(p_2^S)^3$$
$$+ 48(9A-4B)(p_2^S)^2 - 12B(26A-3B)p_2^S + 33AB^2)], \text{ and}$$

$$\underline{r}^S = \frac{1}{8B(16p_2^S - 3B)}[-(4p_2^S - B)\sqrt{K} + B(80(p_2^S)^2 - 12(A+B)p_2^S - 5AB)].$$

Since

$$\frac{12Bp_2^S + 11AB + \sqrt{K}}{32p_2^S} - B = \frac{-20Bp_2^S + 11AB + \sqrt{K}}{32p_2^S} \quad \text{and}$$
$$K - (20Bp_2^S - 11AB)^2 = 128Bp_2^S[2(A-B)p_2^S + AB] > 0,$$

we see that $\underline{p}_1^N + p_2^S - \underline{r}^N = (12Bp_2^S + 11AB + \sqrt{K})/32p_2^S > B$. This completes the proof.   Q.E.D.

Proof of Proposition 4

If $A = B$, then $r^N = r^S = 7A/12$ and we are done. Hence, let $A > B$. From

---

[28] $R = -B^2(6897A^2 - 1288AB + 735B^2)/9$ at $p_2^S = 7B/12$ and $R = -B^2(8349A^2 - 2768AB + 1344B^2)/9$ at $p_2^S = 2B/3$.

the lemma, we have

$$r^N - r^S = \frac{3(4p_2^S - B)\sqrt{K} + 48(8A - 3B)(p_2^S)^2 - 4B(73A - 9B)p_2^S + 33AB^2}{32p_2^S(16p_2^S - 3B)},$$

where

$$K = B[16(16A + 9B)(p_2^S)^2 - 312ABp_2^S + 121A^2B].$$

Since $p_2^S \geq 7B/12$ as shown in the discussion following the lemma, the denominator is positive and so the sign of $r^N - r^S$ is the same as that of the numerator. Note also that $3(4p_2^S - B)\sqrt{K}$ in the numerator is positive. Hence, it suffices to show that

$$[3(4p_2^S - B)\sqrt{K}]^2 - [48(8A - 3B)(p_2^S)^2 - 4B(73A - 9B)p_2^S + 33AB^2]^2$$
$$= 64Ap_2^S(16p_2^S - 3B)[-144(A - B)(p_2^S)^2 + 48B(4A - 3B)p_2^S - B^2(55A - 27B)] > 0$$

or

$$-144(A - B)(p_2^S)^2 + 48B(4A - 3B)p_2^S - B^2(55A - 27B) > 0$$

to prove $r^N > r^S$. Observe that the proof is completed if we can show that the equilibrium price p_2^S lies between the solutions of this quadratic equation, that is, $\underline{p}_2^S < p_2^S < \overline{p}_2^S$, where

$$\underline{p}_2^S = \frac{B\left(8A - 6B - \sqrt{9A^2 - 14AB + 9B^2}\right)}{12(A - B)} \quad \text{and}$$

$$\overline{p}_2^S = \frac{B\left(8A - 6B + \sqrt{9A^2 - 14AB + 9B^2}\right)}{12(A - B)}.$$

To show this, note that it suffices to establish that $f(\underline{p}_2^S) > 0$ and $f(\overline{p}_2^S) < 0$. This is so given the discussion following the lemma on the shape of $f(\cdot)$. For expositional purposes, we let $B = kA$, where $0 \leq k < 1$. Observe that

$$f(\underline{p}_2^S) = \frac{B^2}{6(A-B)^2}[(605A^3 - 802A^2B + 403AB^2 - 126B^3)\sqrt{9A^2 - 14AB + 9B^2}$$

$$-(484A^4 - 151A^3B + 349A^2B^2 - 873AB^3 + 351B^4)]$$

$$= \frac{k^2A^4}{6(1-k)^2}[(605 - 802k + 403k^2 - 126k^3)\sqrt{9 - 14k + 9k^2}$$

$$-(484 - 151k + 349k^2 - 873k^3 + 351k^4)]$$

and

$$f(\overline{p}_2^S) = \frac{B^2}{6(A-B)^2}[-(605A^3 - 802A^2B + 403AB^2 - 126B^3)\sqrt{9A^2 - 14AB + 9B^2}$$

$$-(484A^4 - 151A^3B + 349A^2B^2 - 873AB^3 + 351B^4)]$$

$$= \frac{k^2A^4}{6(1-k)^2}[-(605 - 802k + 403k^2 - 126k^3)\sqrt{9 - 14k + 9k^2}$$

$$-(484 - 151k + 349k^2 - 873k^3 + 351k^4)] \quad .$$

It can be checked that, for all $k \in [0,1)$,

(i) $605 - 802k + 403k^2 - 126k^3 > 0$,

(ii) $484 - 151k + 349k^2 - 873k^3 + 351k^4 > 0$, and

(iii) $[(605 - 802k + 403k^2 - 126k^3)\sqrt{9 - 14k + 9k^2}]^2$

$-[484 - 151k + 349k^2 - 873k^3 + 351k^4]^2 = (1-k)^3 \times$

$[3059969 - 4532055k + 3920906k^2 - 2385990k^3 + 464373k^4 - 19683k^5] > 0$.

This establishes that $f(\underline{p}_2^S) > 0$ and $f(\overline{p}_2^S) < 0$ and we are done.   Q.E.D.

Proof of Proposition 7

(i) Comparison of equilibrium prices

We use p_1^N, p_2^S, r^N, and r^S to denote the equilibrium prices under mixed bundling. The equilibrium prices under no bundling in Proposition 1 are explicitly provided. First of all, recall from the discussion following the lemma that p_2^S is less than $2(1-b)/3 = 2B/3$, which is the equilibrium price of product 2 charged

by firm S under no bundling. Secondly, let us prove that p_1^N is less than $2(1-a)/3 = 2A/3$, which is the equilibrium price of product 1 charged by firm N under no bundling. Observe from part (ii) of the lemma that[29]

$$\frac{2A}{3} - p_1^N = \frac{1}{12B(16p_2^S - 3B)}[X\sqrt{K} + Y] \ ,$$

where

$$X = -3(2p_2^S - B),$$
$$Y = B[72(p_2^S)^2 - 2(17A + 18B)p_2^S + 27AB], \text{ and}$$
$$K = B[16(16A + 9B)(p_2^S)^2 - 312ABp_2^S + 121A^2B].$$

Since $p_2^S \geq 7B/12$, the denominator is positive whereas X is negative. We can also show that $Y > 0$.[30] Hence, the sign of $2A/3 - p_1^N$ is the same as that of

$$Y^2 - X^2K = -8AB(16p_2^S - B)[72(p_2^S)^3 - 108B(p_2^S)^2 + (25A + 36B)Bp_2^S - 15AB^2].$$

Now, by part (i) of the lemma, $(p_2^S)^3$ can be written as

$$(p_2^S)^3 = \frac{24B(37A - 9B)(p_2^S)^2 + 2AB(605A - 387B)p_2^S - 3AB^2(242A - 81B)}{576(A - B)}.$$

Further, if we let $B = kA$, where $0 \leq k < 1$, then $Y^2 - X^2K$ becomes

$$-8kA^3(16p_2^S - 3kA)[24(1 + 27k)(p_2^S)^2 + 2(705 - 343k - 144k^2)Ap_2^S - 3kA^2(282 - 121k)]$$

---

[29] We note that $X, Y, M, N, L, h(\cdot),$ and $\hat{p}$ that appear below are different from those in the proof of the lemma, whereas A, B, K as well as $f(\cdot)$ are the same.

[30] Observe that $Y > 0$ automatically holds if $B \geq (37 - 6\sqrt{30})A/18 \approx 0.22981A$. Otherwise, we can show that $f(\tilde{p}) < 0$, where $\tilde{p} = (17A + 18B - \sqrt{289A^2 - 1332AB + 324B^2})/72$ is the smaller real solution of $Y = 0$. This proves that $p_2^S < \tilde{p}$ and thus $Y > 0$ by the discussion following the lemma on the shape of $f(\cdot)$.

$$= -8kA^3(16p_2^S - 3kA)h(p_2^S),$$

where

$$h(p_2^S) = 24(1 + 27k)(p_2^S)^2 + 2(705 - 343k - 144k^2)Ap_2^S - 3kA^2(282 - 121k).$$

Hence, it suffices to show that $h(p_2^S) < 0$ to prove $2A/3 - p_1^N > 0$. Observe that the sign of the coefficient of $(p_2^S)^2$ in $h(p_2^S)$ is positive but the constant term, $-3kA^2(282 - 121k)$, is negative. Thus, the quadratic equation $h(p) = 0$ has one negative and one positive real solution. Denote the positive solution by $\hat{p}$.[31] The proof is completed if we can show that $p_2^S < \hat{p}$. To show this, note that it suffices to establish that $f(\hat{p}) < 0$. This is so given the discussion following the lemma on the shape of $f(\cdot)$. Observe that

$$f(\hat{p}) = \frac{(1-k)A^4}{6(1+27k)}[M + N\sqrt{L}],$$

where

$$M = -350402625 + 491721285k - 549117517k^2 + 296031583k^3$$
$$- 108880254k^4 + 14350608k^5 + 466560k^6,$$
$$N = 497025 - 465814k + 388711k^2 - 88974k^3 + 3240k^4, \text{ and}$$
$$L = 497025 - 463326k + 454105k^2 - 136440k^3 + 20736k^4.$$

It can be checked that, for all $k \in [0,1)$, we have $M < 0$, $N > 0$, and $M^2 - N^2 L > 0$. This establishes that $f(\hat{p}) < 0$ and we are done.

It remains to show that each firm's bundle price in the equilibrium with mixed bundling is lower than the sum of the stand-alone prices in the equilibrium without bundling. That is, we have to show that

$$r^N < \frac{3 - 2a - b}{3} = \frac{2A + B}{3} \quad \text{and} \quad r^S < \frac{3 - a - 2b}{3} = \frac{A + 2B}{3}.$$

---

[31] $\hat{p} = A(-705 + 343k + 144k^2 + \sqrt{L})/(24(1 + 27k))$, where L is defined below.

The proof of these inequalities is omitted because its steps are almost the same as those used to prove $p_1^N < 2A/3$.[32]

(ii) Comparison of the population of consumers who purchase both products from the same firm

In the equilibrium with mixed bundling, the population $D^{NN} + D^{SS}$ of consumers who purchase both products from the same firm is equal to $1 - D^{NS}$ since there exist no consumers who purchase product 1 from S and product 2 from N, i.e., $D^{SN} = 0$. Thus, to prove that $D^{NN} + D^{SS}$ is larger in the equilibrium with mixed bundling, it suffices to show D^{NS} is smaller in the equilibrium with mixed bundling.
Recall from Figure 3(a) that

$$D^{NS} = (1 - \frac{p_1^N + p_2^S - r^S}{A})(1 - \frac{p_1^N + p_2^S - r^N}{B})$$

under mixed bundling. It is also easy to see from the lemma that, in the equilibrium with mixed bundling, we have

$$p_1^N + p_2^S - r^S = \frac{12(8A + B)q_S - 29AB - \sqrt{K}}{8(16q_S - B)} \quad \text{and}$$

$$p_1^N + p_2^S - r^N = \frac{12Bq_S + 11AB - \sqrt{K}}{32q_S},$$

where

$$K = B[16(16A + 9B)(p_2^S)^2 - 312ABp_2^S + 121A^2B].$$

Using a method similar to that used for the proof of (i), we can show that

$$\frac{p_1^N + p_2^S - r^S}{A} > \frac{1}{3} \quad \text{and} \quad \frac{p_1^N + p_2^S - r^N}{B} > \frac{1}{3}.$$

Hence, we have $D^{NS} < 4/9$ in the equilibrium with mixed bundling. On the

---

[32] Detailed proofs are available upon request.

other hand, Proposition 1 shows that $D^{NS} = 4/9$ in the equilibrium without bundling. This completes the proof.   Q.E.D.

Proof of Proposition 8

It is obvious from part (i) of the lemma that p_2^S converges to 0 as B gets closer to 0. Using this fact, we prove that $\lim_{B \to 0} (p_2^S/B) = 3/5$. Observe that $\lim_{B \to 0} (p_2^S/B) = \lim_{B \to 0} (\partial p_2^S/\partial B)$ by l'Hôpital's rule and that

$$\frac{\partial p_2^S}{\partial B} = -\frac{\partial f(p_2^S)/\partial B}{\partial f(p_2^S)/\partial p_2^S}$$

$$= \frac{576(p_2^S)^3 + 24(37A - 18B)(p_2^S)^2 - 2A(605A - 774B)}{2[864(A - B)(p_2^S)^2 - 24B(37A - 9B)p_2^S - AB(605A - 387B)]}$$

by the Implicit Function Theorem. Since $p_2^S = (p_2^S/B) \cdot B = (\partial p_2^S/\partial B) \cdot B$ for B sufficiently close to 0, we can get the value of $\lim_{B \to 0} (\partial p_2^S/\partial B)$ by inserting $p_2^S = (\partial p_2^S/\partial B) \cdot B$ at p_2^S in the last expression and then letting $B \to 0$ as follows:

$$\lim_{B \to 0} \frac{\partial p_2^S}{\partial B} = \frac{6 - 5 \cdot [\lim_{B \to 0} (\partial p_2^S/\partial B)]}{5} \quad .$$

Hence, $\lim_{B \to 0} (p_2^S/B) = \lim_{B \to 0} (\partial p_2^S/\partial B) = 3/5$. This implies that $p_2^S = 3B/5$ for B sufficiently close to 0. It is now easy to obtain

$$\lim_{B \to 0} p_1^N = \lim_{B \to 0} r^N = \frac{2A}{3}, \quad \text{and} \quad \lim_{B \to 0} r^S = \frac{A}{3}$$

by inserting $p_2^S = 3B/5$ into parts (ii), (iii) and (iv) of the lemma and then letting $B \to 0$.   Q.E.D.

Numerical Results: Tables 5~10

Table 5   Social welfare under mixed bundling

b / a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.8333	0.8419	0.8510	0.8606	0.8708	0.8816	0.8929	0.9049	0.9175	0.9307
	(0.8889)	(0.8944)	(0.9)	(0.9056)	(0.9111)	(0.9167)	(0.9222)	(0.9278)	(0.9333)	(0.9389)
0.1		0.85	0.8586	0.8678	0.8775	0.8879	0.8990	0.9108	0.9232	0.9363
		(0.9)	(0.9056)	(0.9111)	(0.9167)	(0.9222)	(0.9278)	(0.9333)	(0.9389)	(0.9444)
0.2			0.8667	0.8753	0.8846	0.8946	0.9053	0.9167	0.9289	0.9419
			(0.9111)	(0.9167)	(0.9222)	(0.9278)	(0.9333)	(0.9389)	(0.9444)	(0.95)
0.3				0.8833	0.8920	0.9014	0.9117	0.9227	0.9347	0.9475
				(0.9222)	(0.9278)	(0.9333)	(0.9389)	(0.9444)	(0.95)	(0.9556)
0.4					0.9	0.9087	0.9184	0.9289	0.9405	0.9531
					(0.9333)	(0.9389)	(0.9444)	(0.95)	(0.9556)	(0.9611)
0.5						0.9167	0.9255	0.9354	0.9465	0.9588
						(0.9444)	(0.95)	(0.9556)	(0.9611)	(0.9667)
0.6							0.9333	0.9423	0.9526	0.9645
							(0.9556)	(0.9611)	(0.9667)	(0.9722)
0.7								0.95	0.9592	0.9702
								(0.9667)	(0.9722)	(0.9778)
0.8									0.9667	0.9763
									(0.9778)	(0.9833)
0.9										0.9833
										(0.9889)

Table 6   Consumer surplus under mixed bundling

b / a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.0417	0.0764	0.1113	0.1462	0.1813	0.2163	0.2512	0.2859	0.3205	0.3548
	(-0.222)	(-0.161)	(-0.1)	(-0.039)	(0.0222)	(0.0833)	(0.1444)	(0.2056)	(0.2667)	(0.3278)
0.1		0.1375	0.1722	0.2071	0.2421	0.2772	0.3121	0.3469	0.3815	0.4159
		(-0.1)	(-0.039)	(0.0222)	(0.0833)	(0.1444)	(0.2056)	(0.2667)	(0.3278)	(0.3889)
0.2			0.2333	0.2681	0.3030	0.3380	0.3730	0.4079	0.4426	0.4770
			(0.0222)	(0.0833)	(0.1444)	(0.2056)	(0.2667)	(0.3278)	(0.3889)	(0.45)
0.3				0.3292	0.3639	0.3989	0.4339	0.4688	0.5036	0.5381
				(0.1444)	(0.2056)	(0.2667)	(0.3278)	(0.3889)	(0.45)	(0.5111)
0.4					0.425	0.4598	0.4948	0.5298	0.5646	0.5992
					(0.2667)	(0.3278)	(0.3889)	(0.45)	(0.5111)	(0.5722)
0.5						0.5208	0.5556	0.5906	0.6256	0.6602
						(0.3889)	(0.45)	(0.5111)	(0.5722)	(0.6333)
0.6							0.6167	0.6515	0.6865	0.7213
							(0.5111)	(0.5722)	(0.6333)	(0.6944)
0.7								0.7125	0.7474	0.7823
								(0.6333)	(0.6944)	(0.7556)
0.8									0.8083	0.8433
									(0.7556)	(0.8167)
0.9										0.9042
										(0.8778)

Table 7 p_1^N under mixed bundling

b \ a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.58333	0.59183	0.60059	0.60955	0.61861	0.62764	0.63647	0.64498	0.65293	0.66021
0.1		0.525	0.53351	0.54232	0.55131	0.56037	0.56933	0.57797	0.58610	0.59350
0.2			0.46667	0.47519	0.48405	0.49307	0.50210	0.51090	0.51922	0.52677
0.3				0.40833	0.41689	0.42579	0.43484	0.44380	0.45226	0.46003
0.4					0.35	0.35859	0.36755	0.37658	0.38531	0.39327
0.5						0.29167	0.30029	0.30931	0.31825	0.32646
0.6							0.23333	0.24202	0.25106	0.25961
0.7								0.175	0.18377	0.19265
0.8									0.11667	0.12553
0.9										0.05833

Table 8 p_2^S under mixed bundling

b \ a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.58333	0.52930	0.47390	0.41723	0.35941	0.30062	0.24105	0.18096	0.12058	0.06018
0.1		0.525	0.47089	0.41528	0.35827	0.30005	0.24085	0.18093	0.12061	0.06020
0.2			0.46667	0.41246	0.35658	0.29917	0.24049	0.18085	0.12063	0.06022
0.3				0.40833	0.35401	0.29778	0.23989	0.18070	0.12063	0.06024
0.4					0.35	0.29551	0.23885	0.18037	0.12062	0.06026
0.5						0.29167	0.23695	0.17971	0.12053	0.06029
0.6							0.23333	0.17829	0.12025	0.06032
0.7								0.175	0.11942	0.06031
0.8									0.11667	0.06012
0.9										0.05833

Table 9 r^N under mixed bundling

b / a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.66667	0.66202	0.65816	0.65520	0.65324	0.65241	0.65272	0.65433	0.65715	0.66128
0.1		0.6	0.59540	0.59170	0.58899	0.58745	0.58720	0.58828	0.59079	0.59469
0.2			0.53333	0.52878	0.52526	0.52290	0.52191	0.52238	0.52446	0.52810
0.3				0.46667	0.46220	0.45890	0.45701	0.45677	0.45820	0.46155
0.4					0.4	0.39562	0.39267	0.39144	0.39219	0.39503
0.5						0.33333	0.32908	0.32664	0.32638	0.32857
0.6							0.26667	0.26263	0.26097	0.26223
0.7								0.2	0.19632	0.19609
0.8									0.13333	0.13048
0.9										0.06667

Table 10 r^S under mixed bundling

b / a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.66667	0.63166	0.59672	0.56196	0.52749	0.49346	0.45997	0.42714	0.39504	0.36376
0.1		0.6	0.56500	0.53009	0.49539	0.46108	0.42731	0.39420	0.36190	0.33048
0.2			0.53333	0.49833	0.46345	0.42886	0.39476	0.36135	0.32880	0.29720
0.3				0.46667	0.43168	0.39684	0.36240	0.32864	0.29574	0.26395
0.4					0.4	0.36501	0.33026	0.29608	0.26280	0.23072
0.5						0.33333	0.29836	0.26375	0.22999	0.19752
0.6							0.26667	0.23173	0.19739	0.16446
0.7								0.2	0.16513	0.13140
0.8									0.13333	0.09869
0.9										0.06667

References

Adams, W. and Yellen, J. (1976), "Commodity bundling and the burden of monopoly," *Quarterly Journal of Economics,* 90, pp. 475-498.

Anderson, S. and Leruth, L. (1993), "Why firms may prefer not to price discriminate via mixed bundling," *International Journal of Industrial Organization*, 11, pp. 49-61.

Armstrong, M. and Vickers, J. (2010), "Competitive non-linear pricing and bundling," *Review of Economic Studies,* 77, pp. 30-60.

Bakos, Y. and Brynjolfsson, E. (1999), "Bundling information goods: Pricing, profits and efficiency," *Marketing Science,* 45, pp. 1613-1630.

Economides, N. (1993), "Mixed bundling in duopoly," *manuscript.*

Einhorn, M. (1992), "Mix and match compatility with vertical product dimensions," *RAND Journal of Economics*, 23, pp. 535-547.

Gans, J. and King, S. (2006), "Paying for loyalty: Product bundling in oligopoly," *Journal of Industrial Economics,* 54, pp. 43-62.

Kobayashi, B. (2005), "Does economics provide a reliable guide to regulating commodity bundling by firms? A survey of the economic literature," *Journal of Competition Law and Economics,* 1, pp. 707-746.

Matutes, C. and Regibeau, P. (1988), ""Mix and Match": Product compatility without network externalities," *Rand Journal of Economics*, 19, pp. 221-234.

Matutes, C. and Regibeau, P. (1992), "Compatibility and bundling of complementary goods in a duopoly," *Journal of Industrial Economics,* 40, pp. 37-54.

McAfee, P., McMillan, J., and Whinston, M. (1989), "Multiproduct monopoly, commodity bundling and correlation of values," *Quarterly Journal of Economics,* 104, pp. 371-384.

Nalebuff, B. (2004), "Bundling as an entry barrier," *Quarterly Journal of Economics,* 119, pp. 159-187.

Reisinger, M. (2006), "Product bundling and the correlation of valuations in duopoly," *manuscript.*

Schmalensee, R. (1982), "Commodity bundling by a single-product monopolist," *Journal of Law and Economics,* 119, pp. 159-187.

Shaked, A. and Sutton, J. (1983), "Natural oligopolies," *Econometrica*, 51, pp. 1469-1483.

Stole, L. (2007), "Price discrimination and competition," in *Handbook of Industrial Organization: Volume* III, (eds. M. Armstrong and R. Porter), North-Holland, Amsterdam, pp. 2221-2300.

Thanassoulis, J. (2007), "Competitive mixed bundling and consumer surplus," *Journal of Economics & Management Strategy,* 16, pp. 437-467.

Thanassoulis, J. (2011), "Is multimedia convergence to be welcomed?" *Journal of Industrial Economics*, 59, pp. 225-253.

Whinston, M. (1990), "Tying, foreclosure, and exclusion," *American Economic Review,* 80, pp. 837-859.