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On the Optimal Allocation of Prizes in Contests*

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Abstract We characterize the optimal structure of prizes in contests, when the contest designer is interested in the maximization of either the expected total effort or the expected highest effort. The all-pay auction framework in the present paper makes it possible to derive most of the results in Moldovanu and Sela's (2001, *American Economic Review,* 542-558; 2006, *Journal of Economic Theory,* 70–96) incomplete-information model of contests in a particularly simple fashion, as well as to obtain new results.

Keywords Contests, Optimal structure, Prizes, All-pay auctions

JEL Classification D44, J31, D72, D82

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1. INTRODUCTION

How should we design the structure of prizes in contests where many contestants exert irreversible efforts to win a prize? In a very interesting paper, Moldovanu and Sela (2001) constructed an incomplete-information model of contests with multiple, nonidentical prizes in order to answer this question. One of their main results is that, when the contest designer's objective is maximization of expected total effort, (i) it is optimal to award a single prize when cost functions are linear or concave in effort, but (ii) several prizes may be optimal when cost functions are convex.

This paper exploits the well-known fact that contests and all-pay auctions are essentially the same problem and characterizes optimal prize allocation in a particularly simple fashion: The all-pay auction framework in the present paper makes it possible to derive all of the results of Moldovanu and Sela (2001) straightforwardly. Moreover, while they explicitly dealt only with the case of two prizes, the present paper considers more than two prizes without further analytic complications.

This paper also characterizes optimal prize allocation when the contest designer's objective is maximization of expected highest effort. Similar to the case of expected total effort maximization, this paper obtains that (iii) it is optimal to award a single prize when cost functions are linear or concave in effort, but (iv) several prizes may be optimal when cost functions are convex.[1]

Moldovanu and Sela's papers contain an excellent discussion on the optimal prize allocation problem: Readers may consult these papers for motivation and intuition. The present paper aims at demonstrating that the analysis on contests may be facilitated substantially with a proper setup of the problem, as well as at obtaining new results.

2. THE MODEL

Consider a situation where m prizes are awarded to n players. Let $n \geq 2$ and $n \geq m$. Prize k has a 'size' of s^k. Assume without loss of generality that $s^1 \geq s^2 \geq \cdots \geq s^m > 0$ and that $\sum_{k=1}^{m} s^k = 1$. We will call the prize of size s^k as the k-th prize.

[1]We want to mention that Moldovanu and Sela (2006) in another paper considered maximization of highest effort, but they did not provide a proof of Proposition 4 nor did they obtain Propositions 5 and 6 of the present paper. By the way, most of the results in Moldovanu and Sela (2006) can also be easily derived with the present framework.

Player i has a valuation v_i for the prize of size 1. Let x_i denote player i's expenditure to win one of the prizes, and γ_i denote player i's cost parameter. The expenditure may be a monetary bid in auctions, or an effort in contests. Player i's cost of expenditure is given by $\gamma_i c(x_i)$, where $c(\cdot)$ is a strictly increasing function with $c(0) = 0$.[2] Thus, if player i with (v_i, γ_i) exerts x_i, then his payoff is $v_i s^k - \gamma_i c(x_i)$ when he wins the k-th prize while his payoff is $-\gamma_i c(x_i)$ when he wins nothing. Note that this is an unconditional commitment framework in which players have to exert x_i whether or not he wins a prize. The first prize is awarded to the player with the highest expenditure, the second prize is awarded to the player with the second highest expenditure, $\ldots$, and the m-th prize is awarded to the player with the m-th highest expenditure.

Player i's valuation v_i as well as his cost parameter γ_i is private information. Thus, each player's private information is two-dimensional. We now show that we can reduce the dimension of private information. Let p_i^k be the probability that player i wins the k-th prize. Player i's problem is

$$\max_{x_i} v_i \sum_{k=1}^{m} p_i^k s^k - \gamma_i c(x_i).$$

Observe that this problem is equivalent to

$$\max_{x_i} \lambda v_i \sum_{k=1}^{m} p_i^k s^k - \lambda \gamma_i c(x_i)$$

for any $\lambda > 0$. The problem becomes $\max_{x_i} (v_i/\gamma_i) \sum_{k=1}^{m} p_i^k s^k - c(x_i)$ if we set $\lambda = 1/\gamma_i$, and $\max_{x_i} \sum_{k=1}^{m} p_i^k s^k - (\gamma_i/v_i) c(x_i)$ if we set $\lambda = 1/v_i$. In both problems, the private information is one-dimensional as we construct equivalent classes of the form $(v_i, \gamma_i) \sim (v_i/\gamma_i, 1)$ for the former and $(v_i, \gamma_i) \sim (1, \gamma_i/v_i)$ for the latter. Observe that an equivalent class is graphically a ray from the origin. Observe also that the former problem is an all-pay auction problem as in Amann and Leininger (1996), whereas the latter problem is a contest problem as in Moldovanu and Sela (2001, 2006). Thus, as is well-known, all-pay auctions and contests are essentially the same problem. Henceforth, we will normalize $\gamma_i = 1$ for all i, making private information one-dimensional. We assume that each player's valuation is drawn independently from the interval $[\underline{v}, \overline{v}]$ with $0 \leq \underline{v} < \overline{v} \leq \infty$ according to a common distribution F. We assume further that F admits a continuous density function f which is strictly positive on the interval $[\underline{v}, \overline{v}]$.

[2] Moldovanu and Sela (2001, 2006) specified this multiplicative functional form.

Each player chooses his expenditure to maximize the expected payoff, given other players' strategies and prize structure. Hence, player i's strategy is a function $\beta_i : [\underline{v}, \overline{v}] \to \Re_+$ that maps his valuation to the expenditure level. Player i's problem with valuation v_i is

$$\max_{x_i} E_{v_{-i}}[v_i \sum_{k=1}^{m} p_i^k(x_i, \beta_{-i}(v_{-i}))s^k - c(x_i)].$$

In the above expression, we follow the convention that the subscript $-i$ pertains to players other than player i. For example, $v_{-i} = (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

Since players are symmetric, we begin with a heuristic derivation of symmetric equilibrium strategies. Before doing so, observe first that, by setting $s^{m+1} = \cdots = s^n = 0$, we can assume that there always exist n prizes. Next, let $v_{1:n} \geq v_{2:n} \geq \cdots \geq v_{n:n}$ be the order statistics of $v_1, \ldots, v_n$. Note that $v_{k:n}$ is the k-th highest among n valuations drawn from the common distribution F. The distribution and the density of $v_{k:n}$ are denoted by $F_{k:n}$ and $f_{k:n}$, respectively.[3] We also deal with players' valuations except for player i's, so we can similarly have the order statistic $v_{k:n-1}$ and the corresponding functions $F_{k:n-1}$ and $f_{k:n-1}$ of the k-th highest among $n-1$ valuations.

Suppose that players other than i follow a symmetric, increasing and differentiable equilibrium strategy $\beta(\cdot)$. First, it is straightforward to see that player i will never optimally exert an expenditure $x_i > \beta(\overline{v})$. Second, it is also easy to see that a player with valuation $\underline{v}$ will optimally choose an expenditure of zero. Then, player i's expected payoff when his valuation is v_i and he exerts an expenditure of $\beta(w_i)$ is

$$v_i \sum_{k=1}^{n} s^k [F_{k:n-1}(w_i) - F_{k-1:n-1}(w_i)] - c(\beta(w_i))$$

with the convention that $F_{0:n-1}(w_i) \equiv 0$ and $F_{n:n-1}(w_i) \equiv 1$, and (recall that) $s^{m+1} = \cdots = s^n = 0$.

The first-order condition for the payoff maximization with respect to w_i is

$$v_i \sum_{k=1}^{n} s^k [f_{k:n-1}(w_i) - f_{k-1:n-1}(w_i)] - c'(\beta(w_i))\beta'(w_i) = 0,$$

[3] $F_{k:n}(z) = \sum_{r=0}^{k-1} \binom{n}{r} F(z)^{n-r}[1-F(z)]^r$ and $f_{k:n}(z) = \frac{n!}{(k-1)!(n-k)!} F(z)^{n-k}[1-F(z)]^{k-1} f(z)$.

from which we get in equilibrium that

$$
\begin{aligned}
c(\beta(v_i)) &= \sum_{k=1}^{n} s^k \int_{\underline{v}}^{v_i} w[f_{k:n-1}(w) - f_{k-1:n-1}(w)]dw \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) \int_{\underline{v}}^{v_i} w f_{k:n-1}(w)dw.
\end{aligned}
$$

Thus, the equilibrium strategy is

$$
\beta(v_i) = c^{-1}\big(\sum_{k=1}^{n-1} (s^k - s^{k+1}) \int_{\underline{v}}^{v_i} w f_{k:n-1}(w)dw\big).
$$

While this is only a heuristic derivation, it is a standard exercise to show that this is indeed an equilibrium.

3. MAXIMIZATION OF EXPECTED TOTAL EFFORT

Suppose the auctioneer receives all players' expenditures, as in ordinary all-pay auctions. This corresponds to contests in which the contest designer is interested in the expected total sum of players' effort: Moldovanu and Sela (2001) characterized optimal prize allocation for this environment.

3.1. LINEAR COST FUNCTIONS

Assume first that the cost function is linear, i.e., $c(x_i) = x_i$. The auctioneer's revenue is

$$
E\big[\beta(v_{1:n}) + \cdots + \beta(v_{n:n})\big] = \sum_{j=1}^{n} \int_{\underline{v}}^{\overline{v}} \beta(v)dF_{j:n}(v) = n \int_{\underline{v}}^{\overline{v}} \beta(v)dF(v)
$$

$$
\begin{aligned}
&= n \int_{\underline{v}}^{\overline{v}} \sum_{k=1}^{n-1} (s^k - s^{k+1}) \int_{\underline{v}}^{v} w f_{k:n-1}(w)dwdF(v) \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) n \int_{\underline{v}}^{\overline{v}} w f_{k:n-1}(w)[1 - F(w)]dw \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) n \int_{\underline{v}}^{\overline{v}} w \frac{(n-1)!}{(k-1)!(n-1-k)!} F(w)^{n-1-k}[1 - F(w)]^k f(w)dw \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) k \int_{\underline{v}}^{\overline{v}} w \frac{n!}{k!(n-k-1)!} F(w)^{n-k-1}[1 - F(w)]^k f(w)dw
\end{aligned}
$$

$$= \sum_{k=1}^{n-1} (s^k - s^{k+1}) k \int_{\underline{v}}^{\overline{v}} w f_{k+1:n}(w) dw$$
$$= \sum_{k=1}^{n-1} (s^k - s^{k+1}) k \mu_{k+1:n}$$

where $\mu_{k+1:n}$ is the expected value of the $(k+1)$-th highest among n valuations.

What is the optimal structure of prizes in an ordinary all-pay auction? In other words, how should the auctioneer choose $(s^1, \ldots, s^n)$ to maximize her revenue? The following proposition gives the answer.[4]

Proposition 1. *We have $s^1 = 1$ and $s^2 = \cdots = s^n = 0$ in an optimal all-pay auction. That is, the auctioneer's revenue is maximized when she awards only the first prize of size 1.*

Proof. The revenue in an all-pay auction is $\sum_{k=1}^{n-1} (s^k - s^{k+1}) k \mu_{k+1:n} = \sum_{k=1}^{n} \big(k \cdot \mu_{k+1:n} - (k-1)\mu_{k:n}\big) s^k$ with the convention that $\mu_{n+1:n} = 0$. Observe that $\mu_{2:n} > k\mu_{k+1:n} - (k-1)\mu_{k:n}$ for all $k = 2, \ldots, n$ since $\mu_{2:n} > \mu_{k+1:n}$ and $\mu_{k:n} > \mu_{k+1:n}$. Therefore, we have $s^1 = 1$ in an optimal all-pay auction. □

Observe that the auctioneer's revenue $\mu_{2:n}$ in an optimal all-pay auction increases in the number n of players since $v_{k:n} \leq_{lr} v_{k:n'}$, i.e., the former is smaller than the latter in the likelihood ratio order for $n < n'$.

3.2. CONCAVE AND CONVEX COST FUNCTIONS

Assume next that the cost function is either concave or convex. Define $b(\cdot)$ to be the inverse cost function, that is, $b(\cdot) = c^{-1}(\cdot)$. The auctioneer's revenue is

$$R = n \int_{\underline{v}}^{\overline{v}} b\Big(\sum_{k=1}^{n-1} (s^k - s^{k+1}) \int_{\underline{v}}^{v} w f_{k:n-1}(w) dw\Big) dF(v)$$
$$= n \int_{\underline{v}}^{\overline{v}} b\Big(\sum_{k=1}^{n} s^k \int_{\underline{v}}^{v} w[f_{k:n-1}(w) - f_{k-1:n-1}(w)] dw\Big) dF(v),$$

with the convention that $f_{0:n-1}(w) \equiv 0$ and $f_{n:n-1}(w) \equiv 0$. For concave cost functions, we have:

Proposition 2. *Assume that the cost function is concave. We have $s^1 = 1$ and $s^2 = \cdots = s^n = 0$ in an optimal all-pay auction. That is, the auctioneer's revenue is maximized when she awards only the first prize of size 1.*

[4] Moldovanu et al. (2012) provided a similar proof of this proposition.

Proof. For $k = 2, \ldots, n-1$,

$$\frac{\partial R}{\partial s^1} - \frac{\partial R}{\partial s^k} = n \int_{\underline{v}}^{\overline{v}} b'(\cdot) \Big\{ \int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{k-1:n-1}(w) - f_{k:n-1}(w)]dw \Big\} dF(v).$$

Consider $\int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{k-1:n-1}(w) - f_{k:n-1}(w)]dw$. It is zero when $v = \underline{v}$, and $\mu_{1:n-1} + \mu_{k-1:n-1} - \mu_{k:n-1} > 0$ when $v = \overline{v}$. Observe next that

$$\begin{aligned}
&\frac{\partial}{\partial v}\Big(\int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{k-1:n-1}(w) - f_{k:n-1}(w)]dw \Big) \\
&= v\Big\{ (n-1)F(v)^{n-2} f(v) + \frac{(n-1)!}{(k-2)!(n-k)!} F(v)^{n-k}[1-F(v)]^{k-2} f(v) \\
&\quad - \frac{(n-1)!}{(k-1)!(n-k-1)!} F(v)^{n-k-1}[1-F(v)]^{k-1} f(v) \Big\} \\
&= v f(v) F(v)^{n-k-1} \Big\{ (n-1)F(v)^{k-1} + \frac{(n-1)!}{(k-2)!(n-k)!} F(v)[1-F(v)]^{k-2} \\
&\quad - \frac{(n-1)!}{(k-1)!(n-k-1)!} [1-F(v)]^{k-1} \Big\}.
\end{aligned}$$

Let $x = F(v)$. Then, the expression in the brace becomes

$$\begin{aligned}
&(n-1)x^{k-1} + \frac{(n-1)!}{(k-2)!(n-k)!} x(1-x)^{k-2} - \frac{(n-1)!}{(k-1)!(n-k-1)!}(1-x)^{k-1} \\
&= (n-1)x^{k-1} - \frac{(n-1)!}{(k-1)!(n-k)!}(1-x)^{k-2}[(n-k) - (n-1)x].
\end{aligned}$$

When $x = 0$, it is $-(n-k)\frac{(n-1)!}{(k-1)!(n-k)!} < 0$. When $x = 1$, it is $n-1 > 0$. Inspecting the shapes of the first term and the second term, we can easily obtain that there exists $\hat{x} = F(\hat{v})$ with $\hat{v} \in (\underline{v}, \overline{v})$ such that

$$\frac{\partial}{\partial v}\Big(\int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{k-1:n-1}(w) - f_{k:n-1}(w)]dw \Big) > 0$$

if and only if $v > \hat{v}$. This implies that there exists $v^* \in (\underline{v}, \overline{v})$ such that

$$\int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{k-1:n-1}(w) - f_{k:n-1}(w)]dw > 0$$

if and only if $v > v^*$. Since

$$\begin{aligned}
&\int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{k-1:n-1}(w) - f_{k:n-1}(w)]dw dF(v) \\
&= \frac{1}{n}\mu_{2:n} + \frac{k-1}{n}\mu_{k:n} - \frac{k}{n}\mu_{k+1:n} > 0,
\end{aligned}$$

and $b'(\cdot) > 0$ and $b''(\cdot) \geq 0$ due to concavity of the cost function, we conclude that $\partial R/\partial s^1 - \partial R/\partial s^k > 0$ for $k = 2, \ldots, n-1$ because all negative terms are multiplied by relatively low values of $b'(\cdot)$ and vice versa. In addition, it is obvious that $\partial R/\partial s^1 - \partial R/\partial s^n > 0$ since $\partial(\int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{n-1:n-1}(w)]dw)/\partial v$ is nonnegative. Thus, it is optimal to set $s^1 = 1$. □

Next, consider convex cost functions. We have:

Proposition 3. *If*

$$\int_{\underline{v}}^{\overline{v}} b'\Big(\sum_{j=1}^{n} s^j \int_{\underline{v}}^{v} w[f_{j:n-1}(w) - f_{j-1:n-1}(w)]dw\Big) \times \Big\{\int_{\underline{v}}^{v} w[2f_{k-1:n-1}(w) - f_{k-2:n-1}(w) - f_{k:n-1}(w)]dw\Big\} dF(v) < 0$$

for $k = 2, \ldots, n$, then it is not optimal to not award the k-th prize. In particular, if

$$\int_{\underline{v}}^{\overline{v}} b'\Big(\int_{\underline{v}}^{v} wf_{1:n-1}(w)dw\Big)\Big\{2\int_{\underline{v}}^{v} wf_{1:n-1}(w)dw - \int_{\underline{v}}^{v} wf_{2:n-1}(w)dw\Big\} dF(v) < 0,$$

then it is not optimal to award only the first prize, that is, to set $s^1 = 1$.

Proof. Observe that the first and the second inequality in the proposition respectively is equivalent to $\partial R/\partial s^{k-1} - \partial R/\partial s^k < 0$ and $\partial R/\partial s^1 - \partial R/\partial s^2 < 0$, and thus the claim follows. □

Example 1. Let $n = 3$, $F(x) = x$, and $c(x) = x^4$. Calculation shows that

$$\int_{\underline{v}}^{\overline{v}} b'\Big(\int_{\underline{v}}^{v} wf_{1:n-1}(w)dw\Big)\Big\{2\int_{\underline{v}}^{v} wf_{1:n-1}(w)dw - \int_{\underline{v}}^{v} wf_{2:n-1}(w)dw\Big\} dF(v) = -0.0645.$$

Thus, it is optimal to award more than one prize.

4. MAXIMIZATION OF EXPECTED HIGHEST EFFORT

Suppose the auctioneer receives only the highest expenditure. That is, while all players exert expenditures, the auctioneer gets only the highest expenditure but discards all other expenditures. This corresponds to contests in which the contest designer is interested in the expected highest effort. It is often the case that only the best entry matters in many technological research contests for the advancement of human knowledge.

4.1. LINEAR COST FUNCTIONS

When the cost function is linear, i.e., $c(x_i) = x_i$, the auctioneer's revenue is

$$
\begin{aligned}
E[\beta(v_{1:n})] &= \int_{\underline{v}}^{\overline{v}} \sum_{k=1}^{n-1} (s^k - s^{k+1}) \int_{\underline{v}}^{v} w f_{k:n-1}(w) dw dF(v)^n \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) \int_{\underline{v}}^{\overline{v}} w f_{k:n-1}(w)[1-F(w)^n] dw \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) \quad \int_{\underline{v}}^{\overline{v}} w f_{k:n-1}(w)[1-F(w)] \sum_{j=0}^{n-1} F(w)^j dw \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) \sum_{j=0}^{n-1} \int_{\underline{v}}^{\overline{v}} w \frac{(n-1)!}{(k-1)!(n-1-k)!} \cdot \\
&\qquad F(w)^{n-1-k+j}[1-F(w)]^k f(w) dw \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) \sum_{j=0}^{n-1} \frac{(n-1)!}{(n+j)!} \frac{(n-1-k+j)!}{(n-1-k)!} k \int_{\underline{v}}^{\overline{v}} w f_{k+1:n+j}(w) dw \\
&= \sum_{k=1}^{n-1} (s^k - s^{k+1}) \sum_{j=0}^{n-1} \frac{(n-1)!}{(n+j)!} \frac{(n-1-k+j)!}{(n-1-k)!} k \mu_{k+1:n+j}.
\end{aligned}
$$

What is the optimal structure of prizes in this auction? With a proper representation of the revenue given above, we obtain the following proposition.[5]

Proposition 4. *We have $s^1 = 1$ and $s^2 = \cdots = s^n = 0$ in an optimal auction. That is, the auctioneer's revenue is maximized when she awards only the first prize of size 1.*

To prove the proposition, let us define

$$
c^k \equiv \sum_{j=0}^{n-1} \frac{(n-1)!}{(n+j)!} \frac{(n-1-k+j)!}{(n-1-k)!} k \mu_{k+1:n+j}.
$$

Lemma 1. $c^1 > c^k - c^{k-1}$ *for all* $k = 2, \ldots, n-1$.

[5]We note that Moldovanu and Sela (2006, p. 78) stated but not actually proved this proposition. A keen reader would recognize that it is extremely hard, if not impossible, to prove this proposition and perform the ensuing analysis in their original framework.

Proof. Since $\mu_{2:n+j} > \mu_{k+1:n+j}$ and $\mu_{k:n+j} > \mu_{k+1:n+j}$,

$$c^1 + c^{k-1} - c^k > \sum_{j=0}^{n-1} \frac{(n-1)!}{(n+j)!} \mu_{k+1:n+j} \left\{ \frac{(n-2+j)!}{(n-2)!} + (k-1)\frac{(n-k+j)!}{(n-k)!} - k\frac{(n-1-k+j)!}{(n-1-k)!} \right\}.$$

For a given j,

$$\begin{aligned} k\frac{(n-1-k+j)!}{(n-1-k)!} &= k(n-1-k+j)(n-1-k+j-1)\cdots(n-1-k+1) \\ &= (n-1-k+j)(n-1-k+j-1)\cdots(n-1-k+1) \\ &+ (k-1)(n-1-k+j)(n-1-k+j-1)\cdots(n-1-k+1) \\ &< (n-2+j)(n-2+j-1)\cdots(n-2+1) \\ &+ (k-1)(n-k+j)(n-k+j-1)\cdots(n-k+1) \\ &= \frac{(n-2+j)!}{(n-2)!} + (k-1)\frac{(n-k+j)!}{(n-k)!}. \end{aligned}$$

Thus, $c^1 > c^k - c^{k-1}$. □

Since the auctioneer's revenue is $\sum_{k=1}^{n}[c^k - c^{k-1}]s^k$ with the convention that $c^0 = c^n = 0$, it is optimal to set $s^1 = 1$ by the lemma. This proves the proposition.

When $s^1 = 1$, the auctioneer's revenue is $E[\beta(v_{1:n})] = \int_{\underline{v}}^{\overline{v}} w f_{1:n-1}(w)[1 - F(w)^n]dw = \int_{\underline{v}}^{\overline{v}} w f_{1:n-1}(w)dw - \int_{\underline{v}}^{\overline{v}} w f_{1:n-1}(w)F(w)^n dw = \mu_{1:n-1} - \frac{n-1}{2n-1}\mu_{1:2n-1}$, which goes to $\overline{v} - \frac{1}{2}\overline{v} = \frac{1}{2}\overline{v}$ as n goes to infinity. However, the auctioneer's revenue may increase or decrease in the number n of players.

Example 2. Let $F(x) = x^\alpha$. Then, the revenue is

$$\mu_{1:n-1} - \frac{n-1}{2n-1}\mu_{1:2n-1} = \frac{(n-1)\alpha}{(n-1)\alpha+1} - \frac{(n-1)\alpha}{(2n-1)\alpha+1}.$$

When $\alpha = 1$ so that the distribution is uniform, it is $\frac{n-1}{2n}$. Hence, the revenue monotonically increases in n. It is easy to check that the revenue monotonically decreases in n when $\alpha = 6$. It is also easy to check that the revenue first increases and then decreases in n when $\alpha = 4$.

4.2. CONCAVE AND CONVEX COST FUNCTIONS

Assume next that the cost function is either concave or convex. The auctioneer's revenue is

$$\begin{aligned} R &= \int_{\underline{v}}^{\overline{v}} b\Big(\sum_{k=1}^{n-1}(s^k - s^{k+1})\int_{\underline{v}}^{v} w f_{k:n-1}(w)dw\Big)dF(v)^n \\ &= \int_{\underline{v}}^{\overline{v}} b\Big(\sum_{k=1}^{n} s^k \int_{\underline{v}}^{v} w[f_{k:n-1}(w) - f_{k-1:n-1}(w)]dw\Big)dF(v)^n. \end{aligned}$$

For concave functions, we have:

Proposition 5. *Assume that the cost function is concave. We have $s^1 = 1$ and $s^2 = \cdots = s^n = 0$ in an optimal auction. That is, the auctioneer's revenue is maximized when she awards only the first prize of size 1.*

Proof. For $k = 2, \ldots, n-1$,

$$\frac{\partial R}{\partial s^1} - \frac{\partial R}{\partial s^k} = \int_{\underline{v}}^{\overline{v}} b'(\cdot)\Big\{\int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{k-1:n-1}(w) - f_{k:n-1}(w)]dw\Big\}dF(v)^n.$$

The rest of the proof is the same as that of Proposition 2, except we now have to show

$$\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{v} w[f_{1:n-1}(w) + f_{k-1:n-1}(w) - f_{k:n-1}(w)]dw dF(v)^n > 0.$$

Observe that this inequality is in fact $c^1 + c^{k-1} - c^k > 0$, which is true by Lemma 1. □

For convex functions, we have:

Proposition 6. *If*

$$\begin{aligned} &\int_{\underline{v}}^{\overline{v}} b'\Big(\sum_{j=1}^{n} s^j \int_{\underline{v}}^{v} w[f_{j:n-1}(w) - f_{j-1:n-1}(w)]dw\Big) \\ &\times \Big\{\int_{\underline{v}}^{v} w[2f_{k-1:n-1}(w) - f_{k-2:n-1}(w) - f_{k:n-1}(w)]dw\Big\}dF(v)^n < 0 \end{aligned}$$

for $k = 2, \ldots, n$, then it is not optimal to not award the k-th prize. In particular, if

$$\int_{\underline{v}}^{\overline{v}} b'\Big(\int_{\underline{v}}^{v} w f_{1:n-1}(w)dw\Big)\Big\{2\int_{\underline{v}}^{v} w f_{1:n-1}(w)dw - \int_{\underline{v}}^{v} w f_{2:n-1}(w)dw\Big\}dF(v)^n < 0,$$

then it is not optimal to award only the first prize, that is, to set $s^1 = 1$.

Proof. The proof is the same as that of Proposition 3. □

5. CONCLUSION

We have shown that it is optimal to award a single prize when cost functions are linear or concave in effort, but several prizes may be optimal when cost functions are convex. This holds both for the maximization of expected total effort and the maximization of expected highest effort. These results are obtained in an all-pay auction framework, which significantly facilitates the analysis and consequently makes it possible to deal with more than two prizes as well as to characterize the optimal prize structure for the maximization of expected highest effort. It is interesting that the same prize structure is optimal for both maximization problems. We hope that this characterization may cast some light to the real-world contest design situations.

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