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On asymmetry in all-pay auctions*

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Abstract This paper examines the role of asymmetry in all-pay auctions. In particular, this paper decomposes a change in players' valuations into the absolute change and the relative change, and analyzes how these changes affect total expenditures. An increase in the sum of players' valuations increases total expenditures but an increase in asymmetry among players' valuations tends to decrease total expenditures under both complete and incomplete information. This paper also studies the optimal all-pay auction design problem.

Keywords Asymmetry, All-pay auctions, Deterministic contests

JEL Classification C72, D44, D72, D82

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1. INTRODUCTION

All-pay auctions have been used to model such diverse activities as tournaments in job promotions and in sports, R&D races, political campaigns, lobbying, and rent seeking. All-pay auctions, also known as deterministic contests or perfectly discriminating contests, are a particular type of contest in which players exert irreversible expenditures and only the player with the highest expenditure wins the prize.[1]

Players are usually not symmetric in these activities. That is, some players have higher valuations and/or abilities than others. The objective of this paper is to examine how asymmetry among players affects total expenditures or rent dissipation in all-pay auctions. In particular, we decompose a change in players' valuations into the absolute change and the relative change, and analyze how these changes affect total expenditures. Observe that a change in a player's valuation entails two related but conceptually distinct changes: (i) the sum of players' valuations changes and (ii) the ratios among players' valuations change. We call the respective effects of the first and second changes on total expenditures as the absolute effect and the relative effect. As far as we know, this is the first paper that decomposes these two effects and examines the relative effect in particular. In other words, we isolate and examine the relative effect from the total effect.[2]

We first show that, for all-pay auctions under complete information, the absolute effect is always positive whereas the relative effect is always negative. That is, an increase in the sum of players' valuations increases total expenditures but an increase in the asymmetry among players' valuations decreases total expenditures. As for all-pay auctions under incomplete information, the absolute effect continues to be positive but the relative effect is zero when players are initially symmetric and negative when players are (slightly) asymmetric. We also study the optimal all-pay auction design problem and show that the relative effect as well as the absolute effect may be positive.

[1] There is a vast literature on contests. For a relatively recent survey, see Konrad (2009). There are other types of contests, such as Tullock (1980) contests and rank-order tournaments à la Lazear and Rosen (1981).

[2] See the last section for further discussion.

Asymmetric all-pay auctions under complete information have been analyzed rather thoroughly: See Baye *et al.* (1996) and Siegel (2009). In contrast, asymmetric all-pay auctions under incomplete information are very hard to tackle.[3] Amann and Leininger (1996) provided a characterization of equilibrium for two-player asymmetric all-pay auctions, and Kirkegaard (2012, 2013) recently analyzed a tractable two-player asymmetric all-pay auction model. Our analysis on all-pay auctions under incomplete information builds on this model.

The plan of the paper is as follows. Section 2 discusses complete information all-pay auctions, Section 3 discusses incomplete information all-pay auctions, and Section 4 discusses the optimal all-pay auction design problem. Section 5 concludes.

2. ALL-PAY AUCTIONS UNDER COMPLETE INFORMATION

Consider complete information all-pay auctions in which two players exert irreversible expenditures and the player with the highest expenditure wins the prize. Let v_i for $i = 1, 2$ denote player i's valuation for the prize. These valuations are common knowledge. If player i exerts an expenditure x_i, then his payoff is $v_i - x_i$ when he wins the prize and $-x_i$ otherwise. Let $v_1 \geq v_2$ without loss of generality. Then, player 1's expected expenditures, player 2's expected expenditures, and the total expected expenditures in equilibrium are given by, respectively,[4]

$$E_1^c(v_1, v_2) = \frac{v_2}{2},\ E_2^c(v_1, v_2) = \frac{v_2^2}{2v_1},\ \text{and } R^c(v_1, v_2) = \frac{v_2}{2} + \frac{v_2^2}{2v_1}.$$

Observe that $R^c(v_1, v_2)$ decreases when v_1 increases or v_2 decreases. Thus, as asymmetry increases, competitiveness between players and total expenditures (or, rent dissipation) diminish.

Note that a change in v_i entails two kinds of change: the sum $v_1 + v_2$ changes and the ratio between v_1 and v_2 changes. We can decompose any change in

[3] The same is true for other asymmetric auction formats.

[4] See Hillman and Riley (1989) and Baye *et al.* (1996) for detailed equilibrium analysis.

the valuation vector (v_1, v_2) into these changes. To see this, define a measure of (relative) asymmetry $r \equiv v_2/v_1$, and suppose v_1 increases to v_1'. Then, the measure changes from $r = v_2/v_1$ to $r' = v_2/v_1'$. Let us find (v_1'', v_2'') that satisfies (i) $v_1'' + v_2'' = v_1 + v_2$, and (ii) $v_2''/v_1'' = r' = v_2/v_1'$. We get

$$(v_1'', v_2'') = \left(\frac{1}{1+r'}(v_1+v_2), \frac{r'}{1+r'}(v_1+v_2) \right) = \frac{v_1+v_2}{v_1'+v_2}(v_1', v_2).$$

Hence, a change from (v_1, v_2) to (v_1', v_2) can be decomposed into a change from (v_1, v_2) to (v_1'', v_2'') and then a change from (v_1'', v_2'') to (v_1', v_2). The first change is related to the relative effect: the ratio between valuations changes while the sum of valuations remains constant. The second change is related to the absolute effect: observe that (v_1', v_2) is obtained through multiplying the vector (v_1'', v_2'') by the scalar $(v_1' + v_2)/(v_1 + v_2)$ while keeping the ratio r' constant.

Since

$$R^c(v_1'', v_2'') = \frac{v_1+v_2}{v_1'+v_2}\left(\frac{v_2}{2} + \frac{v_2^2}{2v_1'} \right) = \frac{v_1+v_2}{v_1'+v_2} R^c(v_1', v_2) < R^c(v_1', v_2) < R^c(v_1, v_2),$$

we can see that the relative effect decreases total expenditures, and the absolute effect increases total expenditures but with less magnitude.

While the discussion above decomposes a change in one player's valuation into the relative change and the absolute change, we can actually express total expenditures purely in terms of the absolute effect and the relative effect. Observe that, for any (v_1, v_2), we can define the sum $a \equiv v_1 + v_2$ and the ratio $r \equiv v_2/v_1$. Conversely, for any (a, r), we may let $v_1 = a/(1+r)$ and $v_2 = ar/(1+r)$. Thus, (a, r) is a one-to-one transformation of (v_1, v_2). We then have

$$R^c = \frac{v_2}{2}\left(1 + \frac{v_2}{v_1}\right) = \frac{ar}{2}.$$

Thus, the absolute effect is positive whereas the relative effect is negative because

$$\frac{\partial R^c}{\partial a} = \frac{r}{2} > 0, \quad \frac{\partial R^c}{\partial r} = \frac{a}{2} > 0,$$

and r decreases as asymmetry increases.

Summarizing the discussion, we have:

Proposition 1. *For all-pay auctions under complete information, (i) the absolute effect is positive, whereas (ii) the relative effect is negative.*

3. ALL-PAY AUCTIONS UNDER INCOMPLETE INFORMATION

We now consider all-pay auctions under incomplete information. Thus, players' valuations are private information, and they are independently distributed according to the distribution functions F_1 and F_2. The density function f_i for player i is positive on its respective support $[0, \bar{v}_i]$. Assume without loss of generality that $\bar{v}_1 \geq \bar{v}_2$.

For the purpose of demonstration, let us first consider the case when F_i is a uniform distribution on $[0, \bar{v}_i]$ for $i = 1, 2$. As Sahuguet (2006) has shown, the equilibrium strategies for this asymmetric all-pay auction are

$$b_1(v) = \frac{\bar{v}_1 \bar{v}_2}{\bar{v}_1 + \bar{v}_2} \left(\frac{v}{\bar{v}_1}\right)^{\frac{\bar{v}_1+\bar{v}_2}{\bar{v}_1}} \text{ and } b_2(v) = \frac{\bar{v}_1 \bar{v}_2}{\bar{v}_1 + \bar{v}_2} \left(\frac{v}{\bar{v}_2}\right)^{\frac{\bar{v}_1+\bar{v}_2}{\bar{v}_2}}.$$

Thus, players' expected expenditures and total expected expenditures are

$$\begin{aligned}
E_1(\bar{v}_1, \bar{v}_2) &= \int_0^{\bar{v}_1} b_1(v) \frac{1}{\bar{v}_1} dv = \frac{\bar{v}_2}{\bar{v}_1 + \bar{v}_2} \frac{\bar{v}_1^2}{2\bar{v}_1 + \bar{v}_2}, \\
E_2(\bar{v}_1, \bar{v}_2) &= \int_0^{\bar{v}_2} b_2(v) \frac{1}{\bar{v}_2} dv = \frac{\bar{v}_1}{\bar{v}_1 + \bar{v}_2} \frac{\bar{v}_2^2}{\bar{v}_1 + 2\bar{v}_2}, \text{ and} \\
R(\bar{v}_1, \bar{v}_2) &= \frac{\bar{v}_2}{\bar{v}_1 + \bar{v}_2} \frac{\bar{v}_1^2}{2\bar{v}_1 + \bar{v}_2} + \frac{\bar{v}_1}{\bar{v}_1 + \bar{v}_2} \frac{\bar{v}_2^2}{\bar{v}_1 + 2\bar{v}_2}.
\end{aligned}$$

Letting $a \equiv \bar{v}_1 + \bar{v}_2$ and $r \equiv \bar{v}_2 / \bar{v}_1$, we get

$$R = \frac{ar(1 + 4r + r^2)}{(1+r)^2(2+r)(1+2r)}.$$

Hence, $\partial R / \partial a > 0$ and

$$\frac{\partial R}{\partial r} = \frac{2a(1-r)(1 + 8r + 15r^2 + 8r^3 + r^4)}{(1+r)^3(2+r)^2(1+2r)^2} \geq 0$$

for $r \leq 1$. Thus, the absolute effect is positive, and the relative effect is negative (since r decreases as asymmetry increases) except for $r = 1$.

We next turn to more general distributions. We adopt the tractable incomplete information model of Kirkegaard (2012, 2013) and assume that $F_i(v) = F(v/\bar{v}_i)$ for $i = 1, 2$, where F is a distribution whose density function f is positive on its support $[0, 1]$. We assume further, as frequently done in the literature, that the hazard rate (i.e., the failure rate) $f(w)/(1 - F(w))$ is increasing. Note that a distribution F that satisfies this property is said to be IFR. Examples of IFR distributions are exponential, uniform, normal, logistic, power (for $c \geq 1$), Weibull (for $c \geq 1$), and gamma (for $c \geq 1$).[5]

Let $r_i \equiv \bar{v}_j/\bar{v}_i$ denote player i's relative weakness with respect to player j.[6] Thus, the ratio r defined earlier is player 1's relative weakness, i.e., $r_1 = r$ and $r_2 = 1/r$. In addition, let $v_i^s \equiv v_i/\bar{v}_i$ denote player i's "scale-adjusted" valuation and let $k_{ij}^s(v_i^s)$ denote the scale-adjusted valuation of player j that player i with scale-adjusted valuation v_i^s would tie with in the all-pay auction. By substituting the scale-adjusted valuations into Amann and Leininger's (1996) original formula, the function $k_{ij}^s(v_i^s)$ is implicitly defined by

$$\int_{k_{ij}^s(v_i^s)}^1 \frac{f(x)}{x} dx = r_i \int_{v_i^s}^1 \frac{f(x)}{x} dx. \qquad (*)$$

Kirkegaard (2013) showed in his lemma 2 that $E_i(\bar{v}_i, \bar{v}_j) = \bar{v}_i E_s(r_i)$ where

$$E_s(r_i) = r_i \int_0^1 k_{ij}^s(v_i^s)[1 - F(v_i^s)] f(v_i^s) dv_i^s.$$

He also showed that (as suppressing the subscripts for notational simplicity)

$$E_s'(r) = \frac{1}{2}\Big(\int_0^1 k^s(v^s) v^s \Big[\int_{v^s}^1 \frac{f(x)}{v^s} dx - \int_{v^s}^1 \frac{f(x)}{x} dx\Big] f(v^s) dv^s + A(r)\Big),$$

[5]Here, c is the shape parameter.

[6]Note that r_i is lower when player i is stronger than player j, i.e., $\bar{v}_i$ is higher relative to $\bar{v}_j$.

where

$$A(r) = \int_0^1 k^s(v^s)[1-F(v^s)] \int_{v^s}^1 \frac{f(x)}{x} dx \Big(1 - \frac{rf(v^s)}{f(k^s(v^s))}\Big) dv^s.$$

Note that the first term in the expression for $E_s'(r)$ is positive since the difference in the bracket is positive. When $r = 1$, we have $k^s(v^s) = v^s$. Thus, $A(1) = 0$ and so $E_s'(1) > 0$.

Now, we have

$$R = \bar{v}_1 E_s(r_1) + \bar{v}_2 E_s(r_2) = \frac{a}{1+r} E_s(r) + \frac{ar}{1+r} E_s(1/r),$$

using our transformation $\bar{v}_1 = a/(1+r)$ and $\bar{v}_2 = ar/(1+r)$. Hence,

$$\frac{\partial R}{\partial a} = \frac{1}{1+r} E_s(r) + \frac{r}{1+r} E_s(1/r) > 0$$

and so the absolute effect is positive. Next,

$$\frac{\partial R}{\partial r} = \frac{a}{1+r}\Big(E_s'(r) - \frac{1}{r} E_s'(1/r)\Big) - \frac{a}{(1+r)^2}\Big(E_s(r) - E_s(1/r)\Big).$$

It is easy to see that the relative effect is equal to zero when $r = 1$, i.e., when players are symmetric. How about the relative effect for $r < 1$? To see this, we first derive $E_s''(r)$. We have

$$E_s''(r) = \frac{1}{2}\Big(\int_0^1 \frac{\partial k^s(v^s)}{\partial r} v^s \Big[\int_{v^s}^1 \frac{f(x)}{v^s} dx - \int_{v^s}^1 \frac{f(x)}{x} dx\Big] f(v^s) dv^s + A'(r)\Big),$$

where

$$\begin{aligned} A'(r) = &\int_0^1 \frac{\partial k^s(v^s)}{\partial r} [1-F(v^s)] \int_{v^s}^1 \frac{f(x)}{x} dx \Big(1 - \frac{rf(v^s)}{f(k^s(v^s))}\Big) dv^s \\ &+ \int_0^1 k^s(v^s)[1-F(v^s)] \\ &\times \int_{v^s}^1 \frac{f(x)}{x} dx \Big(-\frac{f(v^s)}{f(k^s(v^s))} + \frac{rf(v^s) f'(k^s(v^s)) \frac{\partial k^s(v^s)}{\partial r}}{f(k^s(v^s))^2}\Big) dv^s. \end{aligned}$$

Since (i) $k^s(v^s) = v^s$ when $r = 1$ and (ii) the equation $(*)$ implies

$$\frac{\partial k^s(v^s)}{\partial r} = -\frac{k^s(v^s)}{f(k^s(v^s))}\int_{v^s}^1 \frac{f(x)}{x}dx,$$

we get

$$A'(1) = -\int_0^1 v^s[1-F(v^s)]\int_{v^s}^1 \frac{f(x)}{x}dx\Big(1+\frac{v^s f'(v^s)\int_{v^s}^1 \frac{f(x)}{x}dx}{f(v^s)^2}\Big)dv^s$$

and

$$E_s''(1) = \frac{1}{2}\Big(-\int_0^1 \frac{(v^s)^2}{f(v^s)}\int_{v^s}^1 \frac{f(x)}{x}dx\Big[\int_{v^s}^1 \frac{f(x)}{v^s}dx - \int_{v^s}^1 \frac{f(x)}{x}dx\Big]f(v^s)dv^s + A'(1)\Big).$$

Thus, we see that $E_s''(1) < 0$ if $A'(1) < 0$, or,

$$B \equiv \int_0^1 w(1-F(w))\int_w^1 \frac{f(x)}{x}dx\Big(1+\frac{wf'(w)\int_w^1 \frac{f(x)}{x}dx}{f(w)^2}\Big)dw > 0.$$

We have:

Lemma 1. *$B > 0$ when F is IFR.*

Proof: Let $a(w) \equiv \int_w^1 \frac{f(x)}{x}dx$. If $f'(w) \geq 0$, then obviously $1 + wf'(w)a(w)/f(w)^2 > 0$. If $f'(w) < 0$, then

$$1+\frac{wf'(w)a(w)}{f(w)^2} \geq 1+\frac{1-F(w)}{w}\frac{wf'(w)}{f(w)^2} = \frac{1}{f(w)^2}\Big(f(w)^2+f'(w)(1-F(w))\Big) \geq 0,$$

where the first inequality follows from the fact that $(1-F(w))/w \geq a(w)$ and the second inequality follows from the fact that

$$\frac{d}{dw}\Big(\frac{f(w)}{1-F(w)}\Big) = \frac{f'(w)(1-F(w))+f(w)^2}{(1-F(w))^2} \geq 0$$

since F is IFR. Thus, $B > 0$. *Q.E.D.*

Hence, we have $E_s''(1) < 0$. Next, since

$$\begin{aligned}\frac{\partial^2 R}{\partial r^2} =& \frac{a}{1+r}\Big(E_s''(r) + \frac{1}{r^2}E_s'(1/r) + \frac{1}{r^3}E_s''(1/r)\Big)\\ &- \frac{a}{(1+r)^2}\Big(2E_s'(r) - \frac{1}{r}E_s'(1/r) + \frac{1}{r^2}E_s'(1/r)\Big)\\ &+ \frac{2a}{(1+r)^3}\Big(E_s(r) - E_s(1/r)\Big),\end{aligned}$$

we have $\partial^2 R/\partial r^2 = aE_s''(1) < 0$ when $r = 1$. Recall that $\partial R/\partial r = 0$ when $r = 1$. Therefore, $\partial R/\partial r > 0$, i.e., the relative effect is negative, for r smaller than but sufficiently close to 1.

Summarizing the discussion, we have:

Proposition 2. *For all-pay auctions under incomplete information, (i) the absolute effect is positive, whereas (ii) the relative effect is zero when $r = 1$ and negative for r sufficiently close to 1.*

4. THE OPTIMAL ALL-PAY AUCTION DESIGN

We now study the optimal all-pay auction design problem for this environment. Following Myerson (1981), let $p_i(v_1, v_2)$ denote the probability that player i gets the prize when the valuations are given by v_1 and v_2, and let $q_i(v_i)$ denote the conditional probability that i gets the prize when his valuation is v_i. Since each player exerts an expenditure whether or not he wins the prize, i.e., the expenditure is an unconditional commitment in the terminology of Amann and Leininger (1996), player i's payment depends only on v_i, so let $x_i(v_i)$ denote player i's payment. We additionally require that $p_1(v_1, v_2) + p_2(v_1, v_2) = 1$ for all v_1 and v_2. That is, the prize is always awarded to one of the players.[7] Other than these features, it is a standard exercise to derive an optimal mechanism.

[7] Thus, the mechanism we consider is *constrained* optimal. In a fully optimal mechanism, the prize may sometimes be withheld.

Note first that, for any given $q_i(v_i)$, we have

$$x_i(v_i) = q_i(v_i)v_i - \int_0^{v_i} q_i(s)ds$$

from the definition of player i's expected payoff $U_i(v_i) \equiv q_i(v_i)v_i - x_i(v_i)$ and the well-established fact that $U_i(v_i) = \int_0^{v_i} q_i(s)ds$. Next, let

$$c_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

denote player i's virtual valuation. Note that it is increasing since we assume that F is IFR. In an optimal mechanism, the prize has to be awarded to the player with the highest virtual valuation. Thus, for $i = 1,2$ and $j \neq i$,

$$q_i(v_i) = Pr[c_i(v_i) > c_j(v_j)] = F_j(c_j^{-1}(c_i(v_i))).$$

Let us first consider the case when F_i is a uniform distribution on $[0, \bar{v}_i]$ for $i = 1,2$. Then, we have $c_i(v_i) = 2v_i - \bar{v}_i$. Figure 1 depicts the shapes of the virtual valuations for the uniform case.

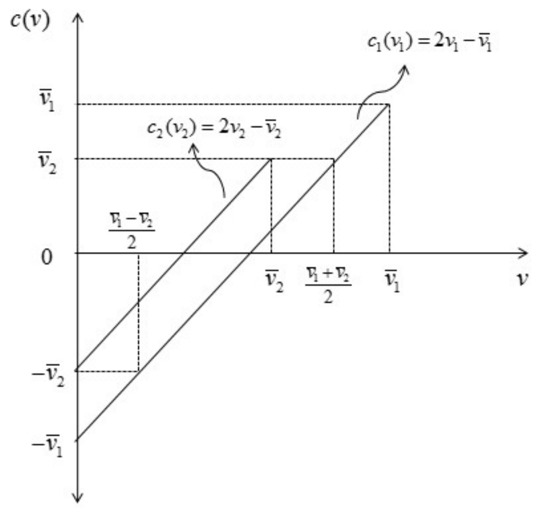

Figure 1: virtual valuations for the uniform distributions

We can derive that

$$q_1(v_1) = \begin{cases} 0 & \text{if } 0 \le v_1 < \frac{\bar{v}_1 - \bar{v}_2}{2}\ ; \\ \frac{2v_1 - \bar{v}_1 + \bar{v}_2}{2\bar{v}_2} & \text{if } \frac{\bar{v}_1 - \bar{v}_2}{2} \le v_1 < \frac{\bar{v}_1 + \bar{v}_2}{2}; \\ 1 & \text{if } \frac{\bar{v}_1 + \bar{v}_2}{2} \le v_1 \le \bar{v}_1, \end{cases}$$

$$q_2(v_2) = \frac{2v_2 + \bar{v}_1 - \bar{v}_2}{2\bar{v}_1} \ \text{ for } \ 0 \le v_2 \le \bar{v}_2.$$

This gives us

$$x_1(v_1) = \begin{cases} 0 & \text{if } 0 \le v_1 < \frac{\bar{v}_1 - \bar{v}_2}{2}; \\ \frac{v_1^2}{2\bar{v}_2} - \frac{(\bar{v}_1 - \bar{v}_2)^2}{8\bar{v}_2} & \text{if } \frac{\bar{v}_1 - \bar{v}_2}{2} \le v_1 < \frac{\bar{v}_1 + \bar{v}_2}{2}; \\ \frac{\bar{v}_1}{2} & \text{if } \frac{\bar{v}_1 + \bar{v}_2}{2} \le v_1 \le \bar{v}_1, \end{cases}$$

$$x_2(v_2) = \frac{v_2^2}{2\bar{v}_1} \ \text{ for } \ 0 \le v_2 \le \bar{v}_2.$$

Thus, players' expected expenditures and total expected expenditures in an optimal mechanism are

$$E_1^*(\bar{v}_1, \bar{v}_2) = \frac{3\bar{v}_1^2 - \bar{v}_2^2}{12\bar{v}_1},\ E_2^*(\bar{v}_1, \bar{v}_2) = \frac{\bar{v}_2^2}{6\bar{v}_1},\ \text{and } R^*(\bar{v}_1, \bar{v}_2) = \frac{3\bar{v}_1^2 + \bar{v}_2^2}{12\bar{v}_1}.$$

Using our transformation of $\bar{v}_1 = a/(1+r)$ and $\bar{v}_2 = ar/(1+r)$, we get

$$R^*(\bar{v}_1, \bar{v}_2) = \frac{a(3+r^2)}{12(1+r)}.$$

Hence, $\partial R^*/\partial a > 0$ and

$$\frac{\partial R^*}{\partial r} = \frac{a(r-1)(r+3)}{12(1+r)^2} \le 0$$

for $r \le 1$. That is, the absolute effect is positive, and the relative effect is also positive (since r decreases as asymmetry increases) except for $r = 1$.

Observe that the sign of the relative effect in the optimal mechanism is opposite to that in the all-pay auction. This is because, as Myerson (1981) already observed, total expenditures may be increased in optimal mechanisms by giving a bid preference (or, favor) to the weak player whose valuation for the prize is lower. Put differently, competition may be made fiercer by leveling the field.

We next turn to more general distributions. Let $t_{ij}(v_i)$ denote the valuation of player j such that $c_i(v_i) = c_j(t_{ij}(v_i))$ holds. That is, $t_{ij}(v_i)$ is the valuation of player j whose virtual valuation is the same as that of player i with valuation v_i. In addition, let $t^s_{ij}(v^s_i)$ denote the scale-adjusted valuation of player j that player i with scale-adjusted valuation v^s_i would tie with in terms of the virtual valuation. Since player i's expected expenditure in any mechanism for a given $q_i(v_i)$ is

$$\int_0^{\bar{v}_i} x_i(v_i) f_i(v_i) dv_i = \int_0^{\bar{v}_i} \left(q_i(v_i)v_i - \int_0^{v_i} q_i(s)ds \right) f_i(v_i) dv_i = \int_0^{\bar{v}_i} c_i(v_i) q_i(v_i) f_i(v_i) dv_i$$

where the second equality obtains by interchanging the order of integration and $q_i(v_i) = F_j(t_{ij}(v_i))$ in an optimal mechanism,

$$E^*_i(\bar{v}_i, \bar{v}_j) = \int_0^{\bar{v}_i} c_i(v_i) F_j(t_{ij}(v_i)) f_i(v_i) dv_i.$$

We have:

Lemma 2. *Player i's expected expenditure in an optimal mechanism can be written as*

$$E^*_i(\bar{v}_i, \bar{v}_j) = \bar{v}_i E^*_s(r_i)$$

where

$$E^*_s(r_i) = \frac{1}{r_i} \int_0^1 v^s_i [1 - F(v^s_i)] \frac{2f(v^s_i)^2 + f'(v^s_i)[1 - F(v^s_i)]}{2f(t^s_{ij}(v^s_i))^2 + f'(t^s_{ij}(v^s_i))[1 - F(t^s_{ij}(v^s_i))]} \frac{f(t^s_{ij}(v^s_i))^3}{f(v^s_i)^2} dv^s_i.$$

Proof: Observe first that

$$\begin{aligned}
E_i^*(\bar{v}_i,\bar{v}_j) &= \int_0^{\bar{v}_i}\Big(v_i-\frac{1-F_i(v_i)}{f_i(v_i)}\Big)F_j(t_{ij}(v_i))f_i(v_i)dv_i\\
&=\bar{v}_i\int_0^{\bar{v}_i}\Big(\frac{v_i}{\bar{v}_i}-\frac{1-F(\frac{v_i}{\bar{v}_i})}{f(\frac{v_i}{\bar{v}_i})}\Big)F\Big(\frac{t_{ij}(v_i)}{\bar{v}_j}\Big)\frac{1}{\bar{v}_i}f\Big(\frac{v_i}{\bar{v}_i}\Big)dv_i\\
&=\bar{v}_i\int_0^{1}\Big(v_i^s-\frac{1-F(v_i^s)}{f(v_i^s)}\Big)F(t_{ij}^s(v_i^s))f(v_i^s)dv_i^s\\
&=\bar{v}_i\int_0^{1}v_i^s[1-F(v_i^s)]\frac{dF(t_{ij}^s(v_i^s))}{dv_i^s}dv_i^s
\end{aligned}$$

where the first equality follows from the definition of $c_i(v_i)$, the second equality follows from $F_i(v_i)=F(v_i/\bar{v}_i)$, the third equality follows from the definition of the scale-adjusted valuation $v_i^s=v_i/\bar{v}_i$ and the fact that $t_{ij}^s(v_i^s)=t_{ij}(v_i)/\bar{v}_j$, and the last equality follows from integration by parts. Note well that we have $t_{ij}^s(v_i^s)=t_{ij}(v_i)/\bar{v}_j$ since $c_j(t_{ij}(v_i))=c_i(v_i)$ and $c_j(\bar{v}_jt_{ij}^s(v_i^s))=c_i(\bar{v}_iv_i^s)$ by definition, and hence $t_{ij}(v_i)=(c_j^{-1}\circ c_i)(v_i)=(c_j^{-1}\circ c_i)(\bar{v}_iv_i^s)=\bar{v}_jt_{ij}^s(v_i^s)$.

We next derive $dF(t_{ij}^s(v_i^s))/dv_i^s$ explicitly. From $c_i(v_i)=c_j(t_{ij}(v_i))$, we have

$$\frac{dt_{ij}(v_i)}{dv_i}=\frac{c_i'(v_i)}{c_j'(t_{ij}(v_i))}=\frac{2f_i(v_i)^2+f_i'(v_i)[1-F_i(v_i)]}{2f_j(t_{ij}(v_i))^2+f_j'(t_{ij}(v_i))[1-F_j(t_{ij}(v_i))]}\frac{f_j(t_{ij}(v_i))^2}{f_i(v_i)^2}.$$

Hence, we can easily derive

$$\frac{dt_{ij}^s(v_i^s)}{dv_i^s}=\frac{\bar{v}_i}{\bar{v}_j}\frac{dt_{ij}(\bar{v}_iv_i^s)}{dv_i}=\frac{1}{r_i}\frac{2f(v_i^s)^2+f'(v_i^s)[1-F(v_i^s)]}{2f(t_{ij}^s(v_i^s))^2+f'(t_{ij}^s(v_i^s))[1-F(t_{ij}^s(v_i^s))]}\frac{f(t_{ij}^s(v_i^s))^2}{f(v_i^s)^2}$$

using the identities $t_{ij}^s(v_i^s)=t_{ij}(\bar{v}_iv_i^s)/\bar{v}_j$ and $F_i(v)=F(v/\bar{v}_i)$. Therefore,

$$\begin{aligned}
dF(t_{ij}^s(v_i^s))dv_i^s =& f(t_{ij}^s(v_i^s))\frac{dt_{ij}^s(v_i^s)}{dv_i^s}\\
=&\frac{1}{r_i}\frac{2f(v_i^s)^2+f'(v_i^s)[1-F(v_i^s)]}{2f(t_{ij}^s(v_i^s))^2+f'(t_{ij}^s(v_i^s))[1-F(t_{ij}^s(v_i^s))]}\frac{f(t_{ij}^s(v_i^s))^3}{f(v_i^s)^2}.
\end{aligned}$$

This gives the desired result. *Q.E.D.*

Now, we have

$$R^* = \bar{v}_1 E_s^*(r_1) + \bar{v}_2 E_s^*(r_2) = \frac{a}{1+r} E_s^*(r) + \frac{ar}{1+r} E_s^*(1/r)$$

using our transformation $\bar{v}_1 = a/(1+r)$ and $\bar{v}_2 = ar/(1+r)$ and so

$$\frac{\partial R^*}{\partial a} = \frac{1}{1+r} E_s^*(r) + \frac{r}{1+r} E_s^*(1/r)$$

and

$$\frac{\partial R^*}{\partial r} = \frac{a}{1+r}\Big(E_s^{*'}(r) - \frac{1}{r} E_s^{*'}(1/r)\Big) - \frac{a}{(1+r)^2}\Big(E_s^*(r) - E_s^*(1/r)\Big).$$

The absolute effect is positive since $E_s^*(r) > 0$ when the distribution F is IFR. (See the proof of Lemma 1.) The relative effect is zero, i.e., $\partial R^*/\partial r = 0$, when $r = 1$. To see the relative effect for $r < 1$, we need to investigate the sign of $E_s^{*''}(1)$ again since $\partial^2 R^*/\partial r^2 = aE_s^{*''}(1)$ when $r = 1$.

From $c_i(v_i) = c_j(t_{ij}(v_i))$, i.e.,

$$v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} = t_{ij}(v_i) - \frac{1 - F_j(t_{ij}(v_i))}{f_j(t_{ij}(v_i))}$$

and $F_i(v_i) = F(v_i/\bar{v}_i)$, we get

$$v_i - \frac{1 - F(v_i/\bar{v}_i)}{f(v_i/\bar{v}_i)/\bar{v}_i} = t_{ij}(v_i) - \frac{1 - F(t_{ij}(v_i)/\bar{v}_j)}{f(t_{ij}(v_i)/\bar{v}_j)/\bar{v}_j}.$$

Since $v_i^s = v_i/\bar{v}_i$ and $t_{ij}^s(v_i^s) = t_{ij}(v_i)/\bar{v}_j$,[8] we have

$$\bar{v}_i\Big(v_i^s - \frac{1 - F(v_i^s)}{f(v_i^s)}\Big) = \bar{v}_j\Big(t_{ij}^s(v_i^s) - \frac{1 - F(t_{ij}^s(v_i^s))}{f(t_{ij}^s(v_i^s))}\Big),$$

or

$$r_i\Big(t_{ij}^s(v_i^s) - \frac{1 - F(t_{ij}^s(v_i^s))}{f(t_{ij}^s(v_i^s))}\Big) = v_i^s - \frac{1 - F(v_i^s)}{f(v_i^s)}.$$

[8]See the proof of Lemma 2.

This gives us

$$\frac{\partial t_{ij}^{s}(v_i^s)}{\partial r_i} = -\frac{c(t_{ij}^{s}(v_i^s))}{r_i c'(t_{ij}^{s}(v_i^s))}.$$

This implies that $E_s^{*''}(r)$ in general involves terms containing f'' and f'''. Therefore, we cannot draw an unambiguous conclusion concerning the sign of $E_s^{*''}(1)$.

Summarizing the discussion, we have:

Proposition 3. *For optimal all-pay auction mechanisms under incomplete information, (i) the absolute effect is positive, whereas (ii) the relative effect may be either positive or negative.*

5. DISCUSSION

We have examined the role of asymmetry in all-pay auctions. We have shown that the absolute effect is always positive under both complete and incomplete information. We have also shown that the relative effect is negative under complete information, but that it is equal to zero when $r = 1$ and negative for r sufficiently close to 1 under incomplete information.

The general analysis under incomplete information was built upon the model of Kirkegaard (2012, 2013) that incorporates asymmetry in a particularly tractable way. Since incomplete information may be modeled in different ways, the results in this paper should be read appropriately. Observe that, when player 1's valuation slightly increases starting from symmetry, we have $\partial R(\bar{v}_1, \bar{v}_2)/\partial \bar{v}_1 > 0$ since $\partial R/\partial a > 0$ and $\partial R/\partial r = 0$ at $r = 1$. This is the main result of Kirkegaard (2013).[9] However, total expected expenditures may decrease in $\bar{v}_1$ when players are sufficiently asymmetric. For instance, when the distribution is uniform, it is straightforward to obtain that $\partial R(\bar{v}_1, \bar{v}_2)/\partial \bar{v}_1 < 0$ when $(\bar{v}_1, \bar{v}_2) = (13, 1)$.

Kirkegaard (2013) is concerned with the total effect, i.e., the sum of the absolute effect and the relative effect, on total expected expenditures. His main point is that the total effect may be positive under incomplete information, which contrasts with the fact that the total effect is negative under complete informa-

[9] See Proposition 1 of Kirkegaard (2013).

tion. In comparison, our main point is that the relative effect is negative under both complete and incomplete information. We have also analyzed the optimal all-pay auction design problem. The main finding is that the relative effect is positive for the uniform distribution but may be either positive or negative for general distributions.

It would be desirable to extend the analysis of asymmetry in all-pay auctions to the case of more than two players. But, this seems to be a difficult task since even the equilibrium characterizations are elusive as Parreiras and Rubinchik (2010) have demonstrated.

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